ON THE CORRELATION SCALE OF STOCHASTIC FIELDS

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CONTENTS

1. INTRODUCTION

The followings are well known in the second order analysis of stochastic process theory: (1) a homogeneous stochastic process is usually characterized by its mean value and correlation function, (2) the correlation function represents the variance and correlation structure of the process, (3)the correlation function is related to the power spectral density function by means of the Wiener Khintchine transform pair, and (4) if the process is Gaussian, then all its characteristics are known only from its mean value and correlation function or power spectral density function. Therefore, when the stochastic process theory is applied to the analysis of observed field data, a set of these observations is interpreted as a realization of a homogeneous stochastic process. Then, the mean value and the correlation function or the power spectral density function are usually estimated following routes 1 and 2 shown schematically in Fig. 1.1. Finally, the resulting correlation function and power spectral density function are in general summarized in analytical forms.

In the above procedures usually encountered in practical field data analyses, the last step of modeling is, of course, based on not only the observed data, but also physical understanding of the phenomena and engineering judgement. Hence, the modeling task cannot be successfully achieved without understanding the phenomena indicated by the observed data and without taking into account the accuracy required for analyses.

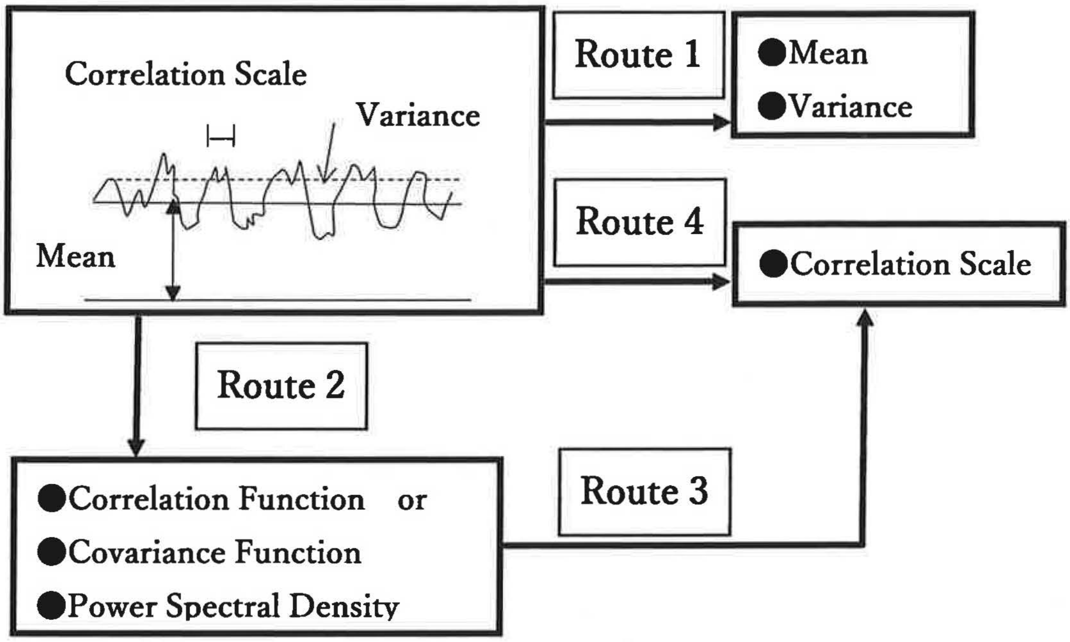


Fig. 1-1 Schematic Diagram Showing Relationship Between Stochastic Process and Its Statistics

In addition to the correlation function, however, if simple statistics (correlation scale) could be defined that are able to represent the correlation structure of the stochastic processes and also can be directly estimated from a set of observed field data without the correlation function or the power spectral density function, these statistics for correlation could provide quite useful information for capturing the essential phenomena indicated by the observed data and eventually in the modeling of its stochastic process. Consequently, it is asserted that three statistics (the mean, variance and correlation scale) could be used as the fundamental parameters to approximately characterize stochastic processes.

In fact, as briefly described in Section 1.1, instead of the correlation function, the correlation scales summarized in Table 1.1 have been successfully used as the measures of correlation structure of stochastic processes in studies of the turbulence, signal analysis and the stochastic response analysis of mechanical systems to dynamic loading. However, since these correlation scales are defined through the correlation function, they cannot be evaluated before knowing the correlation function. From a viewpoint of the statistical analysis of stochastic process data, it is desirable to estimate the correlation scale directly from observed data without going through the correlation function. Hence, the definitions for the correlation scales indicated in Table 1.1 are quite useless from the point of view of the statistical analysis of stochastic process data.

In this study, the two new definitions for the correlation scales A and C in Table 1.1 are discussed which are suitable for statistical analysis of observed data in the sense described above. Hence, the problem dealt with in this study is to develop a practical procedure for estimating correlation scales (Route 4 in Fig. 1.1). To do this, the variance behavior of an averaging process previously studied by Panchev (1971), Bendat and Piersol (1971) and Vanmarcke (1983) is analysed in a systematic way. The procedure used in this study is especially similar to that used by Vanmarcke (1983). However, the results and their interpretation are quite different from those of previous studies, and the two different definitions for correlation scales are reinterpreted in a consistent way from a viewpoint of the statistical analysis of stochastic process data. In this study, a practical procedure utilizing a graphical method as occasion demands, is presented to estimate the correlation scales from observed data. The procedure for one dimensional stochastic process data is also extended to the two dimensional case and the significance of the correlation scales for two dimensional stochastic process is briefly discussed using numerically generated two dimensional stochastic fields. In the final chapter, some new application examples of correlation scales are described.

Table 1-1 Summary of Definitions for The Scale of Correlation

Items	Definitions	Authors
A	$\frac{1}{\sigma^2}\int_{-\infty}^{\infty} R(\xi)d\xi$	Taylor (1935), Batchelor (1953), Tatarski (1961), Monin and Yaglom (1965), Panchev (1971), Bendat and Piesol (1971), Lumley (1970), Vanmarche (1983)
B	$\frac{1}{\sigma^2}\int_{-\infty}^{\infty} \mid R(\xi) \mid d\xi$	Stratonovich (1967)
C	$\sqrt{-\frac{2R(0)}{R''(0)}}$	Tatarski (1961), Monin and Yaglom (1965), Lumley (1970), Harada and Shinozuka (1979)
D	$\dfrac{\int_{-\infty}^{\infty} \xi \mid R(\xi) \mid d\xi}{\int_{-\infty}^{\infty} \mid R(\xi) \mid d\xi}$	Lin, Fujimori and Ariaratnam (1979)
Note	$\sigma^2 = R(0)$: Variance, $R(\xi)$: Correlation Function, $R''(\xi) = d^2R(\xi) / d\xi^2$	

1.1 Brief Historical Note on Correlation Scales

Table 1.1 summarizes the definitions of correlation scales in the literature available. In the study of turbulence, Taylor (1935) first proposed a measure of the correlation scale to obtain low variance estimates of the mean value of fluctuating velocities. The ratio of a finite sampling interval to the correlation (A in Table 1.1) is used as the equivalent number of independent observations from stochastic process data. Batchelor (1953), Tatarski (1961), and Monin and Yaglom (1965) also used the same measure proposed by Taylor (1921) in their studies of isotropic turbulence. In the study of random signal analysis (Panchev (1971), Bendat and Piersol (1971)), the correlation scale A in Table 1.1 was also used for the condition of ergodicity with respect to the mean value. Stratonovitch (1967) used the other definition B as indicated in Table 1.1 in the discussion of the condition of ergodicity with respect to the mean value by considering the averaging process. A correlation scale C in Table 1.1 was proposed to represent the inner scale of turbulence (Tatarski (1961), Monin and Yaglom (1965), and Lumley (1970)).

In the study of stochastic response of mechanical systems to dynamic loading (Lin at al. (1979)), the other definition of correlation scale D in Table 1.1 was used. This correlation scale is proposed in such a way that if the correlation scale (time) of dynamic loading is much smaller than the relaxation time of the mechanical system, the response can be approximated by a Markov process. Thus, many convenient mathematical properties related to the Markov processes can be used to solve the system response to random dynamic loading (Statonovitch (1967), Lin (1979) and Wu (1985)).

Recently, Vanmarcke (1983) reinterpreted the correlation scale in A in Table 1.1 from the viewpoint of the analysis of the variance of averaging processes in a manner similar to the discussion of Stratonovitch (1963) and Pancheve (1971), and represented many applications in civil and mechanical engineering problems. Harada and Shinozuka (1985) recently proposed the correlation scale C in Table 1.1 in their analysis of the spatial variations of seismic ground motions by considering the variance of difference processes.

In conclusion, the previous definitions for correlation scales were all based on the correlation function or the power spectral density function and tend to vague in why they are defined as shown in Table 1.1, except the studies of Vanmarcke, and Harada and Shinozuka. Thus, to obtain the correlation scale, the correlation function or the power spectral density function has to be given first. This kind of definition is not useful from the viewpoint of the statistical analysis of observed data because it is desirable to estimate the correlation scale directly from the observed data without using the correlation function.

2. VARIANCE OF AVERAGING PROCESS AND DIFFERENCE PROCESS

Since any continuous parameter homogeneous stochastic process with mean m and variance σ_{ff}^2 can be expressed as the sum of its mean and homogeneous stochastic process $f(x)$ with zero mean and variance σ_{ff}^2, we consider a homogeneous stochastic process $f(x)$ with zero mean and variance σ_{ff}^2 in the analysis that follows.

For a homogeneous stochastic process $f(x)$, a family of the averaging process $f_D(x)$ may be defined such that,

$$f_D(x) = \frac{1}{D} \int_{x-D/2}^{x+D/2} f(y)dy \tag{2.1}$$

Introducing the following indefinite integral $F(x)$ of $f(x)$,

$$F(x) = \int_{-\infty}^{x} f(y)dy \qquad \text{or} \qquad \frac{dF(x)}{dx} = f(x) \tag{2.2}$$

Equation (2.1) is also written as,

$$f_D(x) = \frac{1}{D} F_D(x) \tag{2.3}$$

where,

$$F_D(x) = F\left(x + \frac{D}{2}\right) - F\left(x - \frac{D}{2}\right) \tag{2.4}$$

The function $F_D(x)$ is the finite difference process of $F(x)$. In Eqs. (2.1) and

(2.3), the averaging process $f_D(x)$ and difference process $F_D(x)$ are always homogeneous since the original process $f(x)$ is homogeneous. However, the indefinite integral process $F(x)$ is not always homogeneous. The condition for the homogeneity of $F(x)$ is very closely related to the behavior of the power spectral density function $S_{ff}(\kappa)$ of $f(x)$ at origin $\kappa = 0$.

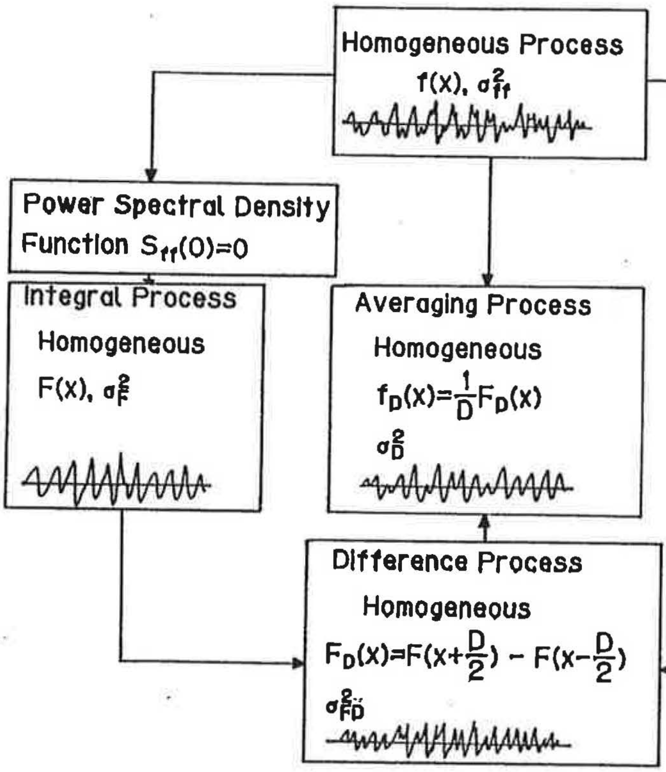


Fig. 2.1 Schematic Diagram Showing Relationships Among Integral Process, Averaging Process and Difference Process.

Figure 2.1 summarizes the above conclusion for the integral process $F(x)$, averaging process $f_D(x)$ and difference process $F_D(x)$. Explanation using equations are shown in below.

If $F(x)$ is homogeneous, the power spectral density function $S_{FF}(\kappa)$ of $F(x)$ is well known to be expressed due to Eq. (2.2) as,

$$S_{FF}(\kappa) = \frac{S_{ff}(\kappa)}{\kappa^2} \tag{2.5}$$

As is also well known, $S_{ff}(\kappa)$ is in turn related to the correlation function $R_{ff}(\xi)$ through the Winer-Kintchine transform pair,

$$\begin{aligned} S_{ff}(\kappa) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{ff}(\xi) e^{-i\kappa\xi} d\xi \\ R_{ff}(\xi) &= \int_{-\infty}^{\infty} S_{ff}(\kappa) e^{i\kappa\xi} d\kappa \end{aligned} \tag{2.6a}$$

Accounting for the symmetry of $R_{ff}(\xi)$ with respect to the origin ($R_{ff}(\xi) = R_{ff}(-\xi)$), the Winer-Kintchine transform pair is also given such as,

$$S_{ff}(\kappa) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{ff}(\xi) \cos \kappa\xi d\xi \tag{2.6b}$$

Using the asymptotic expansion of $\cos \kappa\xi$, $S_{ff}(\kappa)$ can be expressed as,

$$\begin{aligned} S_{ff}(\kappa) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{ff}(\xi) \left(\sum_{n=0}^{\infty} (-1)^n \frac{(\kappa\xi)^{2n}}{(2n)!} \right) d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{ff}(\xi) d\xi - \frac{1}{2\pi} \frac{\kappa^2}{2!} \int_{-\infty}^{\infty} \xi^2 R_{ff}(\xi) d\xi + \cdots \end{aligned} \tag{2.7}$$

Then, from Eqs. (2.5) and (2.7), $S_{FF}(\kappa)$ is also expressed as,

$$S_{FF}(\kappa) = \frac{1}{2\pi} \frac{1}{\kappa^2} \int_{-\infty}^{\infty} R_{ff}(\xi) d\xi - \frac{1}{2\pi} \frac{1}{2!} \int_{-\infty}^{\infty} \xi^2 R_{ff}(\xi) d\xi + \cdots \tag{2.8}$$

Therefore, if $S_{ff}(0) = (1/2\pi)\int_{-\infty}^{\infty} R_{ff}(\xi)d\xi \neq 0$, then $S_{FF}(\kappa)$ is singular at the origin. This means that the variance of $F(x)$ becomes infinity and the process $F(x)$ is no longer homogeneous.

It should be noted again that the difference process $F_D(x)$ of $F(x)$ given by Eq. (2.4) is always homogeneous even in the case where the process $F(x)$ is nonhomogeneous because the averaging process $F_D(x)$ is always homogeneous (see Eq. (2.1)). More rigorous discussion concerning the homogeneity of the integral and difference processes can be seen in the followings: Cramer and Leadbetter (1967), Doob (1953), and Yaglom (1962, 1973).

Turning to the variance σ_D^2 of the averaging process $f_D(x)$, we first consider the power spectral density function $S_{f_D}(\kappa)$ of $f_D(x)$. $S_{f_D}(\kappa)$ is given as follows:

$$S_{f_D}(\kappa) = \left(\frac{\sin \kappa D/2}{\kappa D/2}\right)^2 S_{ff}(\kappa) \tag{2.9}$$

Above equation is derived from the following general well-known equations in the filtering theory (for example, Papoulis (1984)),

$$f_D(x) = \frac{1}{D}\int_{x-D/2}^{x+D/2} f(y)dy = \int_{-\infty}^{\infty} f(y)g(x-y)dy$$

$$g(y) = \begin{cases} 1/D & -D/2 \leq y \leq D/2 \\ 0 & \text{otherwise} \end{cases} \tag{2.10a}$$

where $g(y)$ is the impulse response function of a system, and the power spectral density function of the response $f_D(x)$ to the input $f(x)$ is given by

$$S_{f_D}(\kappa) = | G(\kappa) |^2 S_{ff}(\kappa) \tag{2.10b}$$

in which $G(\kappa)$ is the transfer function of the system that is related to $g(y)$ such as,

$$\begin{aligned} G(\kappa) &= \int_{-\infty}^{\infty} g(y) \mathrm{e}^{-i\kappa y} dy = \frac{1}{D} \int_{-D/2}^{D/2} \mathrm{e}^{-i\kappa y} dy \\ &= \frac{\sin(\kappa D / 2)}{(\kappa D / 2)} \end{aligned} \tag{2.10c}$$

Substitution of Eq. (1.10c) into Eq. (2.10b) yields Eq. (2.9).

Utilizing the following relationship between the basic spectral window and the triangle window,

$$\begin{aligned} &\left(\frac{\sin \kappa D / 2}{\kappa D / 2} \right)^2 = \frac{1}{D} \int_{-\infty}^{\infty} \left(1 - \frac{|\xi|}{D} \right) \cos \kappa \xi d\xi \\ &\text{where,} \\ &1 - \frac{|\xi|}{D} = \begin{cases} 1 - \dfrac{|\xi|}{D} & |\xi| \leq D \\ 0 & |\xi| > D \end{cases} \end{aligned} \tag{2.11}$$

The variance σ_D^2 of $f_D(x)$ is given following its definition such that,

$$\begin{aligned} \sigma_D^2 &= E\left[f_D^2(x) \right] = R_{f_D}(0) = \int_{-\infty}^{\infty} S_{f_D}(\kappa) d\kappa \\ &= \int_{-\infty}^{\infty} \left(\frac{\sin \kappa D / 2}{\kappa D / 2} \right)^2 S_{ff}(\kappa) d\kappa \end{aligned} \tag{2.12a}$$

or,

$$\begin{aligned}\sigma_D^2 &= \frac{1}{D}\int_{-\infty}^{\infty}\left(1-\frac{|\xi|}{D}\right)\left(\int_{-\infty}^{\infty} S_{ff}(\kappa)\cos\kappa\xi d\kappa\right)d\xi \\ &= \frac{1}{D}\int_{-\infty}^{\infty}\left(1-\frac{|\xi|}{D}\right)R_{ff}(\xi)d\xi\end{aligned} \qquad (2.12b)$$

If $F(x)$ is homogeneous (this means $S_{ff}(0)=0$), Eq. (2.12a) is also expressed in terms of the correlation function $R_{FF}(\xi)$ of the indefinite integral process $F(x)$ such as,

$$\begin{aligned}\sigma_D^2 &= \int_{-\infty}^{\infty}\left(\frac{\sin\kappa D/2}{\kappa D/2}\right)^2 S_{ff}(\kappa)d\kappa \\ &= \left(\frac{2}{D}\right)^2\int_{-\infty}^{\infty}\frac{S_{ff}(\kappa)}{\kappa^2}\sin^2\left(\frac{\kappa D}{2}\right)d\kappa \\ &= \left(\frac{2}{D}\right)^2\int_{-\infty}^{\infty}\frac{S_{ff}(\kappa)}{\kappa^2}\sin^2\left(\frac{\kappa D}{2}\right)d\kappa \\ &= \left(\frac{2}{D}\right)^2\int_{-\infty}^{\infty}S_{FF}(\kappa)\left(\frac{1-\cos\kappa D}{2}\right)d\kappa \\ &= \frac{2}{D^2}\left(R_{FF}(0)-R_{FF}(D)\right)\end{aligned} \qquad (2.13)$$

Summarizing Eqs. (2.12a), (2.12b) and (2.13), we obtained,

$$\begin{aligned}\sigma_D^2 &= \int_{-\infty}^{\infty}\left(\frac{\sin\kappa D/2}{\kappa D/2}\right)^2 S_{ff}(\kappa)d\kappa \\ &= \frac{1}{D}\int_{-\infty}^{\infty}\left(1-\frac{|\xi|}{D}\right)R_{ff}(\xi)d\xi\end{aligned} \qquad (2.14a)$$

and, when $S_{ff}(0)=0$ ($F(x)$ is homogeneous),

$$\sigma_D^2 = \frac{2}{D^2}\left(R_{FF}(0)-R_{FF}(D)\right) \qquad (2.14b)$$

The variance $\sigma_{F_D}^2$ of the difference process $F_D(x)$ is related to σ_D^2 due to Eq. (2.3) as follow,

$$\sigma_{F_D}^2 = D^2\sigma_D^2 \rightarrow \sigma_D^2 = \frac{1}{D^2}\sigma_{F_D}^2 \tag{2.15}$$

The relationships between the processes $f(x), f_D(x), F(x)$ and $F_D(x)$ may also summarized schematically as shown in Fig. 2.1. In the case where the power spectral density function at the origin $S_{ff}(0) \neq 0$, the indefinite integral process $F(x)$ of $f(x)$ is nonhomogeneous, but otherwise $F(x), F_D(x)$ and $f_D(x)$ are all homogeneous stochastic processes with variances $\sigma_{FF}^2, \sigma_{F_D}^2$ and σ_D^2, respectively. Equations (2.14) and (2.15) play a fundamental role in the new interpretation of the definition of the correlation scale which is capable of estimating directly from the observation data without recourse to the correlation function.

3. DEFINITION AND SIGUNIFICANCE OF CORRELATION SCALE

We consider in this chapter the behavior of the variance σ_D^2 of $f_D(x)$ in the two limiting cases where $D \to 0$ and $D \to \infty$, using Eqs. (2.14) and (2.15). And then, we introduce new definitions for the correlation scales.

In the first case where $D \to 0$, using Eq. (2.14a) together with the relationship of $\sin \kappa D / 2 = \kappa D / 2$ for $D \to 0$, we can easily show that,

$$\sigma_D^2 = \int_{-\infty}^{\infty} S_{ff}(\kappa) d\kappa = \sigma_{ff}^2 \qquad \text{as } D \to 0 \tag{3.1}$$

If $S_{ff}(0) = 0$, then the integral process $F(x)$ is homogeneous. Hence, we can consider Eq. (2.14b). The correlation function $R_{FF}(D)$ can be expanded into a Taylor series around $D = 0$ such that,

$$R_{FF}(D) = R_{FF}(0) + R'_{FF}(0)D + \frac{1}{2!}R''_{FF}(0)D^2 + \cdots \tag{3.2a}$$

in which $R'_{FF}(0) = dR_{FF}(\xi) / d\xi \,|_{\xi=0}$ and similar definitions apply to $R''_{FF}(0)$, etc. And also, the correlation function $R_{FF}(\xi)$ and its derivations are given by,

$$\begin{aligned} R_{FF}(\xi) &= \int_{-\infty}^{\infty} S_{FF}(\kappa) \cos \kappa\xi d\kappa \\ R'_{FF}(\xi) &= -\int_{-\infty}^{\infty} \kappa S_{FF}(\kappa) \sin \kappa\xi d\kappa \\ R''_{FF}(\xi) &= -\int_{-\infty}^{\infty} \kappa^2 S_{FF}(\kappa) \cos \kappa\xi d\kappa = -\int_{-\infty}^{\infty} S_{ff}(\kappa) \cos \kappa\xi d\kappa \end{aligned} \tag{3.2b}$$

Therefore, $R'_{FF}(0) = 0$. Then, using Eq. (3.2a), Eq. (2.14b) can be written as,

$$\begin{aligned}\sigma_D^2 &= \frac{2}{D^2}\left(R_{FF}(0) - R_{FF}(D)\right) \qquad \text{as } D \to 0 \\ &= -R''_{FF}(0) = \int_{-\infty}^{\infty} S_{ff}(\kappa)d\kappa = \sigma_{ff}^2\end{aligned} \tag{3.3}$$

In the second case where $D \to \infty$, using the last equation of Eq. (2.14a), we can easily obtain,

$$\sigma_D^2 = \frac{1}{D}\int_{-\infty}^{\infty} R_{ff}(\xi)d\xi \qquad \text{as } D \to \infty \tag{3.4a}$$

If $S_{ff}(0) = 0$, from Eq. (2.14b), we obtain,

$$\sigma_D^2 = \frac{2}{D^2}R_{FF}(0) = \frac{2}{D^2}\sigma_{FF}^2 \qquad \text{as} \quad D \to \infty \tag{3.4b}$$

At this point, we define the variance κ_F^2 of wavenumber of integral process $F(x)$ such as,

$$\kappa_F^2 = \frac{\int_{-\infty}^{\infty} \kappa^2 S_{FF}(\kappa)d\kappa}{\int_{-\infty}^{\infty} S_{FF}(\kappa)d\kappa} = \frac{\int_{-\infty}^{\infty} S_{ff}(\kappa)d\kappa}{\int_{-\infty}^{\infty} S_{FF}(\kappa)d\kappa} = \frac{\sigma_{ff}^2}{\sigma_{FF}^2} \tag{3.5a}$$

The standard deviation κ_F can be interpreted as a predominant wavenumber of the integral process $F(x)$ of wavenumber domain, then the predominant wavelength $L_F = 2\pi / \kappa_F$ of $F(x)$ can be given using Eqs. (3.3) and (3.5a) such as,

$$L_F = 2\pi \frac{\sigma_{FF}}{\sigma_{ff}} = 2\pi \sqrt{-\frac{R_{FF}(0)}{R''_{FF}(0)}} = 2\pi \sqrt{\frac{\int_{-\infty}^{\infty} S_{FF}(\kappa)d\kappa}{\int_{-\infty}^{\infty} S_{ff}(\kappa)d\kappa}} \tag{3.5b}$$

By using the above predominant wave length, the Eq. (3.4b) is expressed as,

$$\sigma_D^2 = \frac{2}{D^2}\left(\frac{L_F}{2\pi}\right)^2 \sigma_{ff}^2 \qquad \text{as} \quad D \to \infty \tag{3.6}$$

In summary of the above expressions of variance σ_D^2 of the averaging process $f_D(x)$ for $D \to 0$ and $D \to \infty$,

Case I ($S_{ff}(0) \neq 0$):

$$\frac{\sigma_D}{\sigma_{ff}} = \begin{cases} 1 & ,D \to 0 \\ \dfrac{1}{\sqrt{D}}\sqrt{\dfrac{\int_{-\infty}^{\infty} R_{ff}(\xi)d\xi}{\sigma_{ff}^2}} & ,D \to \infty \end{cases} \tag{3.7a}$$

Case II ($S_{ff}(0) = 0$):

$$\frac{\sigma_D}{\sigma_{ff}} = \begin{cases} 1 & ,D \to 0 \\ \dfrac{1}{D}\left(\dfrac{L_F}{\pi\sqrt{2}}\right) & ,D \to \infty \end{cases} \tag{3.7b}$$

The above Eqs. (3.7a) and (3.7b) provide incentives of introducing the two scales of correlation L^*, L_F^* such as,

$$L^* = \frac{\int_{-\infty}^{\infty} R_{ff}(\xi)d\xi}{\sigma_{ff}^2} = 2\pi \frac{S_{ff}(0)}{\int_{-\infty}^{\infty} S_{ff}(\kappa)d\kappa}, \qquad \text{for } S_{ff}(0) \neq 0$$

$$L_F^* = \frac{L_F}{\pi\sqrt{2}} = \frac{1}{\pi\sqrt{2}}\left(2\pi\frac{\sigma_{FF}}{\sigma_{ff}}\right) \tag{3.8}$$

$$= \frac{1}{\pi\sqrt{2}}\left(2\pi\sqrt{\frac{\int_{-\infty}^{\infty} S_{FF}(\kappa)d\kappa}{\int_{-\infty}^{\infty} S_{ff}(\kappa)d\kappa}}\right), \qquad \text{for } S_{ff}(0) = 0$$

Utilizing the above scales of correlation, Eqs. (3.7a) and (3.7b) are written as a simple expression,

Case I ($S_{ff}(0) \neq 0$):

$$\frac{\sigma_D}{\sigma_{ff}} = \begin{cases} 1 & ,D \to 0 \\ \sqrt{\dfrac{L^*}{D}} & ,D \to \infty \end{cases} \tag{3.9a}$$

Case II ($S_{ff}(0) = 0$):

$$\frac{\sigma_D}{\sigma_{ff}} = \begin{cases} 1 & ,D \to 0 \\ \dfrac{L_F^*}{D} = \dfrac{\sqrt{2}}{D}\dfrac{\sigma_{FF}}{\sigma_{ff}} & ,D \to \infty \end{cases} \tag{3.9b}$$

Although the scales of correlation L^*, L_F^* defined by Eq. (3.8) possess the same forms as A and C in Table 1.1, which are defined by previous investigators, the significance of the correlation scales in this study is quite clear as follows.

The correlation scale L_F^* defined by the second equation of Eq. (3.8) is such that when the averaging distance D reaches the distance of the correlation scale L_F^*, the standard deviation σ_D becomes $\sqrt{2}\sigma_{FF} / D$ according to Eq. (3.9b). This is the same standard deviation of σ_D when $F(x+D)$ and $F(x)$ become completely uncorrelated as $D \to \infty$ (see Eqs. (2.14b) and (3.9b)).

Also, for the correlation scale L^* defined by the first equation of Eq. (3.8), a similar consideration can be made using Eq. (3.9a) as follows. When the averaging distance D reaches the distance of the correlation scale L^*, the standard deviation σ_D becomes σ_{ff} in accordance with Eq. (3.9a). This is the same standard deviation of the averaging process $f_D(x)$ and the original process $f(x)$ may be considered to be a perfectly correlated process, i.e., $R_{ff}(D) = R_{f_D}(D) = \sigma_{ff}^2$ for $D \to 0$ (see Eqs. (2.14a) and (3.9a)).

The definitions of the correlation scales L^* and L_F^* are also interpreted on terms of the wavenumber as follows: Since the wavenumber κ is related to the wavelength L such that $\kappa = 2\pi / L$, we may define the (wavenumber) spectral scales κ^* and κ_F^* corresponding to the correlation scales (correlation distances) L^* and L_F^*, respectively such as,

$$\kappa^* = \frac{2\pi}{L^*} \quad \text{and} \quad \kappa_F^* = \frac{2\pi}{L_F^*} \tag{3.10}$$

Then, Eq. (3.10) can be written using Eqs. (3.5b) and (3.8) together with the

Wiener-Kintchine relationship given by Eq. (2.6) as follows,

$$\kappa^* = \frac{1}{S_{ff}(0)} \int_{-\infty}^{\infty} S_{ff}(\kappa) d\kappa$$

$$\kappa_F^* = \pi\sqrt{2} \sqrt{\frac{\int_{-\infty}^{\infty} S_{ff}(\kappa) d\kappa}{\int_{-\infty}^{\infty} S_{FF}(\kappa) d\kappa}} = \pi\sqrt{2}\, \kappa_F \tag{3.11}$$

where κ_F (see Eq. (3.5a) is the apparent (predominant) wavenumber of the power spectral density function $S_{FF}(\kappa)$ of indefinite integral process $F(x)$.

The significance of the spectral scales defined by Eq. (3.11) is illustrated in Fig. 3.1. It may be observed from Fig. 3.1 that the spectral scales represent a large wavenumber above which the power spectral density function may be considered to be zero.

As demonstrated in the numerical example, the definitions of correlation scales given by Eqs. (3.8) and (3.9) are also useful for estimation the correlation scales from the observed data since the standard deviations σ_{ff} and σ_D or σ_{FF} can be easily calculated by following their definitions from the observed data.

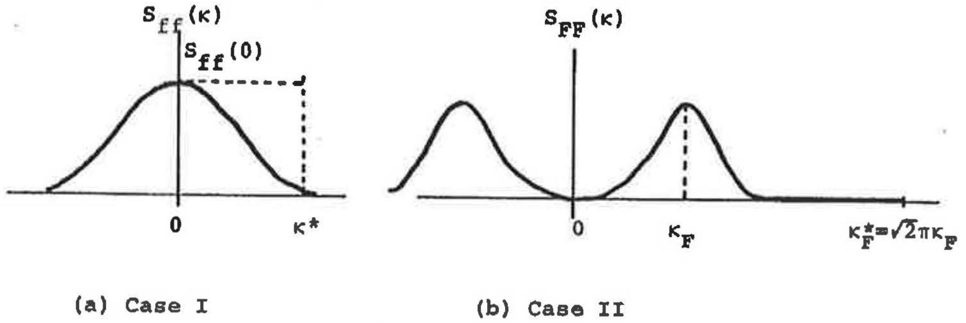


Fig. 3.1 Schematical Illustration of Significance of the Spectral Scales ((a) Case I and (b) Case II)

4. EXAMPLES OF CORRELATION FUNCTION, POWER SPECTRAL DENSITY FUNCTION AND THEIR CORRELATION SCALES

Before the practical estimation method and numerical examples using the graphical representation of the correlation scales (Chapter 5,6), we present here the several examples of correlation functions, or power spectral density functions, and their correlation scales according to their definitions of Chapter 3.

4.1 Example of Correlation Functions, Power Spectral Density Functions, and Correlation Scale for Case I ($S_{ff}(0) \neq 0$)

Table 4.1-1 and Fig. 4.1-1 show the example of models of normalized correlation functions and power spectral density functions for Case I ($S_{ff}(0) \neq 0$).

The correlation scales for these models can be obtained from the definition by the upper equation of Eq. (3.8) as follows for Table 4.1-1;

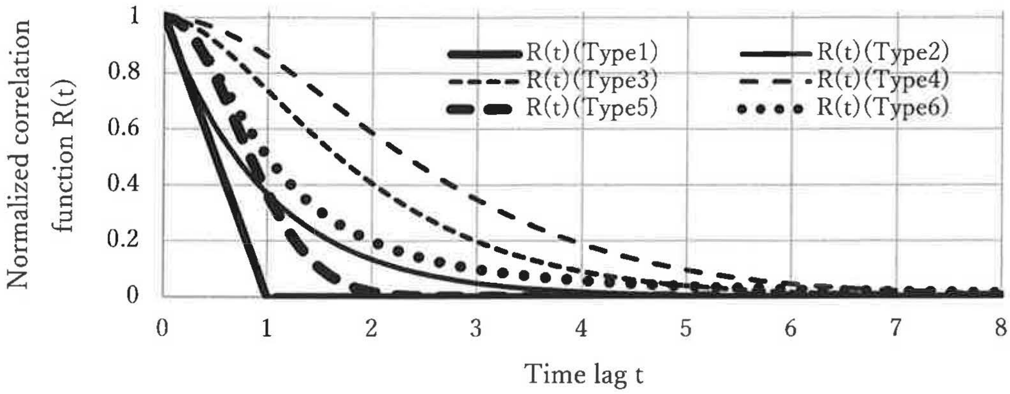
Type 1: $L^* = 2\pi \dfrac{S_{ff}(0)}{\sigma_{ff}^2} = 2\pi \dfrac{b}{2\pi} = b$ Type 2 : $L^* = 2\pi \dfrac{S_{ff}(0)}{\sigma_{ff}^2} = 2\pi \dfrac{b}{\pi} = 2b$

$$\text{Type 3}: \quad L^* = 2\pi \frac{S_{ff}(0)}{\sigma_{ff}^2} = 2\pi \frac{2b}{\pi} = 4b \quad \text{Type 4}: \quad L^* = 2\pi \frac{S_{ff}(0)}{\sigma_{ff}^2} = 2\pi \frac{8b}{3\pi} = \frac{16}{3} b$$

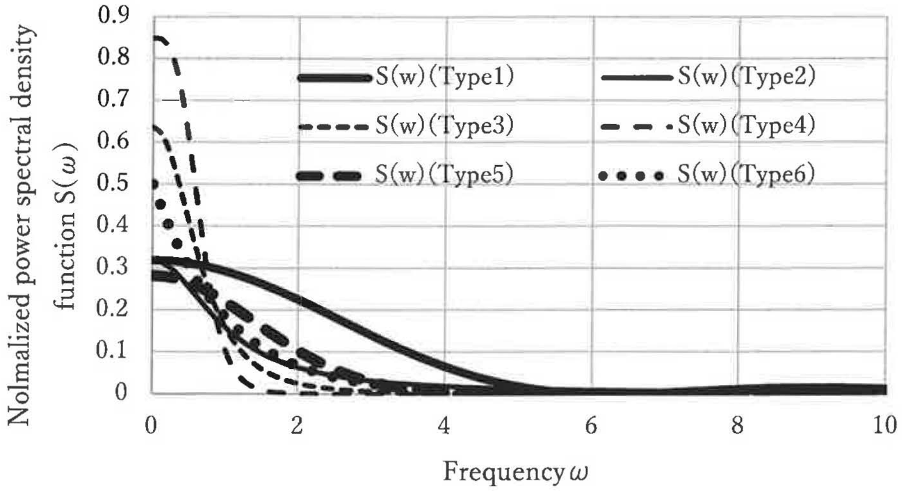
$$\text{Type 5}: \quad L^* = 2\pi \frac{S_{ff}(0)}{\sigma_{ff}^2} = 2\pi \frac{b}{2\sqrt{\pi}} = b\sqrt{\pi} \quad \text{Type 6}: \quad L^* = 2\pi \frac{S_{ff}(0)}{\sigma_{ff}^2} = 2\pi \frac{b}{2} = \pi b$$

Table 4.1-1 Example of Models of Normalized Correlation Functions and Power Spectral Density Functions for Case I ($S_{ff}(0) \neq 0$)

Type	$R(\tau)\,/\,\sigma^2, \sigma^2 = R(0)$	$S(\omega)\,/\,\sigma^2$
1	$1 - \frac{\lvert\tau\rvert}{b} \quad \lvert\tau\rvert \le b$ $0 \quad$ otherwise	$\frac{1}{\pi b} \frac{(1-\cos b\omega)}{\omega^2}$
2	$e^{-(\frac{\lvert\tau\rvert}{b})}$	$\frac{b}{\pi} \frac{1}{1+(b\omega)^2}$
3	$\left(1 + \frac{\lvert\tau\rvert}{b}\right) e^{-(\frac{\lvert\tau\rvert}{b})}$	$\frac{2b}{\pi} \frac{1}{\left(1+(b\omega)^2\right)^2}$
4	$\left(1 + \frac{\lvert\tau\rvert}{b} + \frac{1}{3}(\frac{\lvert\tau\rvert}{b})^2\right) e^{-(\frac{\lvert\tau\rvert}{b})}$	$\frac{8b}{3\pi} \frac{1}{\left(1+(b\omega)^2\right)^3}$
5	$e^{-(\frac{\lvert\tau\rvert}{b})^2}$	$\frac{b}{2\sqrt{\pi}} e^{-(\frac{b\omega}{2})^2}$
6	$\frac{1}{1+(\frac{\tau}{b})^2}$	$\frac{1}{2} b e^{-b\lvert\omega\rvert}$



(a) Correlation Functions for Case I ($S_{ff}(0) \neq 0$) in Table 4.1-1



(b) Power Spectral Density Functions for Case I ($S_{ff}(0) \neq 0$) in Table 4.1-1

($2S(\omega)$ for only Type1 is shown)

Fig. 4.1-1 Diagrams of Example of Models of Normalized Correlation Functions and Power Spectral Density Functions for Case I ($S_{ff}(0) \neq 0$) in Table 4.1-1

Two examples of exact behavior of variance ratio $\sigma_D^2 / \sigma_{ff}^2$ using Type 1 and Type2 of Table 4.4-1 are shown as follows,

Substituting the normalized correlation function of Type 1 of Table 4.4-1 to Eq. (2.14a), $\sigma_D^2 / \sigma_{ff}^2$ is given such as,

$$\frac{\sigma_D^2}{\sigma_{ff}^2} = \frac{2}{D}\int_0^D \left(1-\frac{\xi}{D}\right)\frac{R_{ff}(\xi)}{\sigma_{ff}^2}d\xi$$
$$= \frac{2}{D}\begin{cases} \displaystyle\int_0^D \left(1-\frac{\xi}{D}\right)\left(1-\frac{\xi}{b}\right)d\xi & D \le b \\ \displaystyle\int_0^b \left(1-\frac{\xi}{D}\right)\left(1-\frac{\xi}{b}\right)d\xi & D > b \end{cases} \tag{4.1-1a}$$

The above integration in conjunction with the correlation scale gives the exact behavior of $\sigma_D^2 / \sigma_{ff}^2$,

$$\frac{\sigma_D^2}{\sigma_{ff}^2} = \begin{cases} 1-\dfrac{D}{3L^*} & D \le b = L^* \\ \left(\dfrac{L^*}{D}\right)\left(1-\dfrac{L^*}{3D}\right) & D > b = L^* \end{cases} \tag{4.1-1b}$$

The above equation gives for the two extreme cases,

$$\frac{\sigma_D^2}{\sigma_{ff}^2} = \begin{cases} 1 & D \to 0 \\ \dfrac{L^*}{D} & D \to \infty \end{cases} \tag{4.1-1c}$$

Similarly, for Type 2 of Table 4.4-1, $\sigma_D^2 / \sigma_{ff}^2$ is given by,

$$
\begin{aligned}
\frac{\sigma_D^2}{\sigma_{ff}^2} &= \frac{2}{D}\int_0^D \left(1-\frac{\xi}{D}\right) \mathrm{e}^{-\xi/b} d\xi \\
&= 2\left(\frac{b}{D}\right)^2 \left(\frac{D}{b} - 1 + \mathrm{e}^{-\xi/b}\right) \\
&= \left(\frac{L^*}{D}\right)\left(1 - \frac{1}{2}\left(\frac{L^*}{D}\right)\left(1-\mathrm{e}^{-2D/L^*}\right)\right), \quad L^* = 2b
\end{aligned}
\tag{4.1-2a}
$$

In deriving above equation, a following indefinite integration was used.

$$
\int x\mathrm{e}^{ax} dx = \frac{\mathrm{e}^{ax}}{a^2}\left(ax-1\right) \tag{4.1-2b}
$$

Using the Eq. (4.4-2a) gives for the two extreme cases,

$$
\frac{\sigma_D^2}{\sigma_{ff}^2} = \begin{cases} 1 & D \to 0 \\ \dfrac{L^*}{D} & D \to \infty \end{cases} \tag{4.1-2c}
$$

In derivation above equation, the following approximate equation was used.

$$
\mathrm{e}^{-x} = \begin{cases} 1 - x + \dfrac{1}{2!}x^2 - \dfrac{1}{3!}x^3 \cdots & x \to 0 \\ 0 & x \to \infty \end{cases} \tag{4.1-2d}
$$

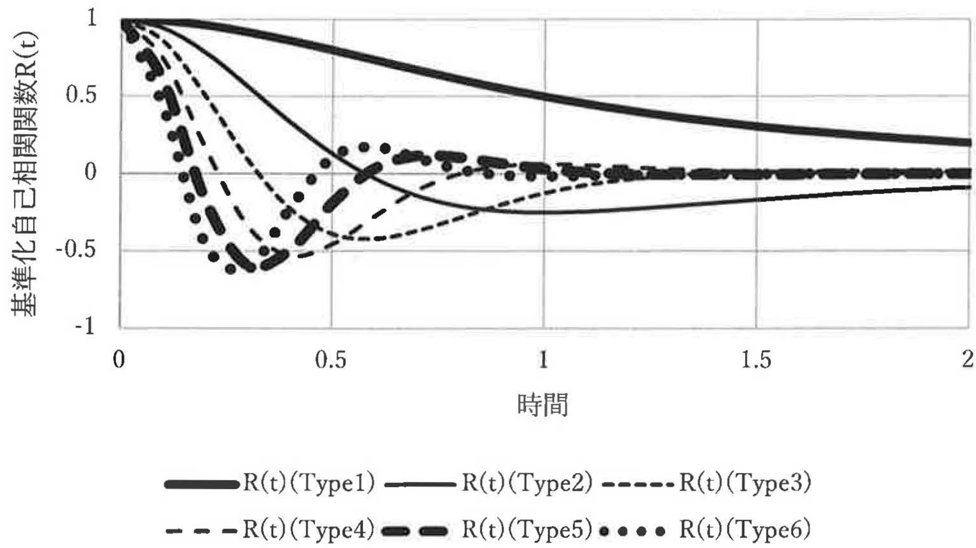
4.2 Example of Correlation Functions, Power Spectral Density Functions, and Correlation Scale for Case II ($S_{ff}(0) = 0$)

Table 4.2-1, Table 4.2-2 and Fig. 4.2-1, Fig. 4.2-2 show the example of models of normalized correlation functions and power spectral density functions for Case II ($S_{ff}(0) = 0$).

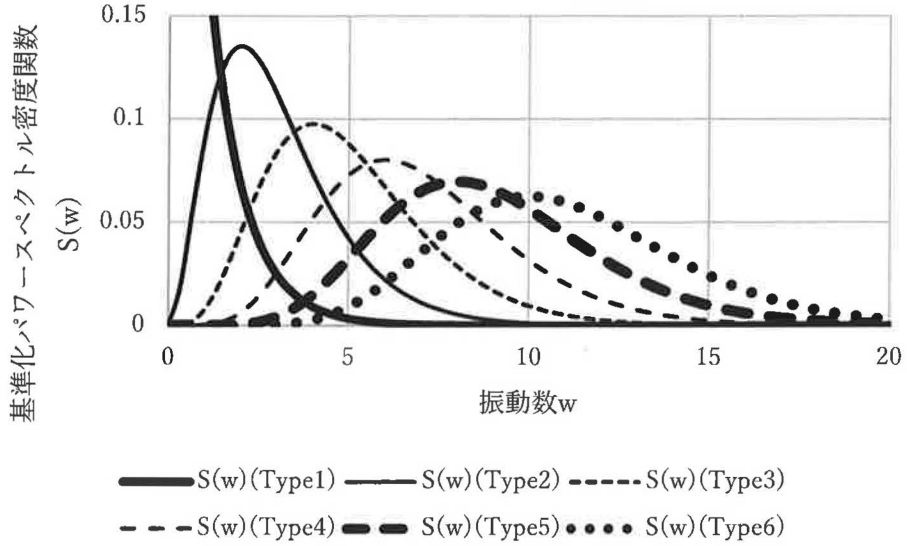
Table 4.2-1 Example of Models of Normalized Correlation Functions and Power Spectral Density Functions for Case II ($S_{ff}(0)=0$)

(Type 1 is Case I)

Type	$R(\tau)\,/\,\sigma^2, \sigma^2 = R(0)$	$S(\omega)\,/\,\sigma^2$
1	$\dfrac{b^2}{b^2+\tau^2}$	$\dfrac{1}{2\cdot 0!} b e^{-b\|\omega\|}$
2	$\dfrac{b^4(b^2-3\tau^2)}{(b^2+\tau^2)^3}$	$\dfrac{1}{2\cdot 2!} b^3\omega^2 e^{-b\|\omega\|}$
3	$\dfrac{b^6(b^4-10b^2\tau^2+5\tau^4)}{(b^2+\tau^2)^5}$	$\dfrac{1}{2\cdot 4!} b^5\omega^4 e^{-b\|\omega\|}$
4	$\dfrac{b^8(b^6-21b^4\tau^2+35b^2\tau^4-7\tau^6)}{(b^2+\tau^2)^7}$	$\dfrac{1}{2\cdot 6!} b^7\omega^6 e^{-b\|\omega\|}$
5	$\dfrac{b^{10}(b^8-36b^6\tau^2+12b^4\tau^4-84b^2\tau^6+9\tau^8)}{(b^2+\tau^2)^9}$	$\dfrac{1}{2\cdot 8!} b^9\omega^8 e^{-b\|\omega\|}$
6	$\dfrac{b^{12}\begin{pmatrix} b^{10}-55b^8\tau^2+330b^6\tau^4-462b^4\tau^6+ \\ 165b^2\tau^8-11\tau^{10} \end{pmatrix}}{(b^2+\tau^2)^{11}}$	$\dfrac{1}{2\cdot 10!} b^{11}\omega^{10} e^{-b\|\omega\|}$



(a) Correlation Functions for Case II ($S_{ff}(0) = 0$) in Table 4.2-1 (Type I is Case I)



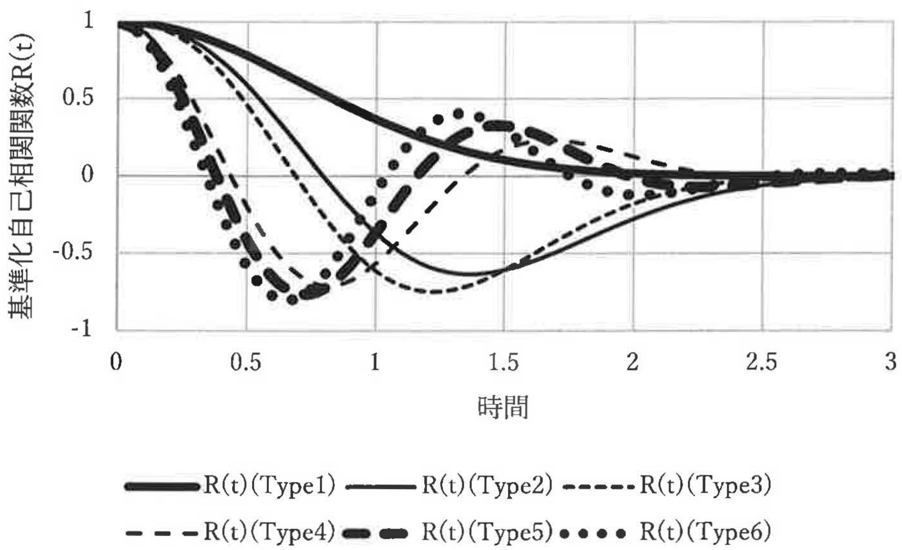
(b) Power Spectral Density Functions for Case II ($S_{ff}(0) = 0$) in Table 4.2-1 (Type I is Case I)

Fig. 4.2-1 Diagrams of Example of Models of Normalized Correlation Functions and Power Spectral Density Functions for Case II ($S_{ff}(0) = 0$) in Table 4.2-1

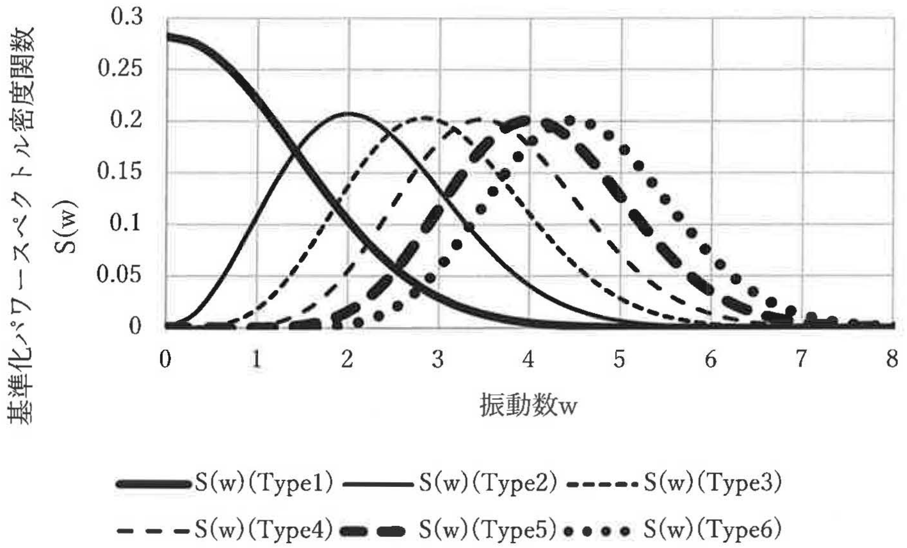
Table 4.2-2 Example of Models of Normalized Correlation Functions and Power Spectral Density Functions for Case II ($S_{ff}(0) = 0$)

(Type I is Case I)

Type	$R(\tau) / \sigma^2, \sigma^2 = R(0)$	$S(\omega) / \sigma^2$
1	$e^{-(\frac{\tau}{b})^2}$	$\frac{b}{2\sqrt{\pi}} e^{-(\frac{b\omega}{2})^2}$
2	$\left(1 - 2(\frac{\tau}{b})^2\right) e^{-(\frac{\tau}{b})^2}$	$\frac{1}{2} \cdot \frac{b^3}{2\sqrt{\pi}} \omega^2 e^{-(\frac{b\omega}{2})^2}$
3	$\left(1 - 4(\frac{\tau}{b})^2 + \frac{4}{3}(\frac{\tau}{b})^4\right) e^{-(\frac{\tau}{b})^2}$	$\frac{1}{2} \cdot \frac{b^5}{12\sqrt{\pi}} \omega^4 e^{-(\frac{b\omega}{2})^2}$
4	$\left(1 - 6(\frac{\tau}{b})^2 + 4(\frac{\tau}{b})^4 - \frac{8}{15}(\frac{\tau}{b})^6\right) e^{-(\frac{\tau}{b})^2}$	$\frac{1}{2} \cdot \frac{b^7}{120\sqrt{\pi}} \omega^6 e^{-(\frac{b\omega}{2})^2}$
5	$\left(1 - 8(\frac{\tau}{b})^2 + 8(\frac{\tau}{b})^4 - \frac{32}{15}(\frac{\tau}{b})^6 + \frac{16}{105}(\frac{\tau}{b})^8\right) e^{-(\frac{\tau}{b})^2}$	$\frac{1}{2} \cdot \frac{b^9}{1680\sqrt{\pi}} \omega^8 e^{-(\frac{b\omega}{2})^2}$
6	$\left(1 - 10(\frac{\tau}{b})^2 + \frac{40}{3}(\frac{\tau}{b})^4 - \frac{16}{3}(\frac{\tau}{b})^6 + \frac{16}{21}(\frac{\tau}{b})^8 - \frac{32}{945}(\frac{\tau}{b})^{10}\right) e^{-(\frac{\tau}{b})^2}$	$\frac{1}{2} \cdot \frac{b^{11}}{30240\sqrt{\pi}} \omega^{10} e^{-(\frac{b\omega}{2})^2}$



(a) Correlation Functions for Case II ($S_{ff}(0) = 0$) in Table 4.2-2
(Type I is Case I)



(b) Power Spectral Density Functions for Case II ($S_{ff}(0) = 0$) in Table 4.2-2
(Type I is Case I)

Fig. 4.2-2 Diagrams of Example of Models of Normalized Correlation Functions and Power Spectral Density Functions for Case II ($S_{ff}(0) = 0$) in Table 4.2-2

The correlation scales for these models can be obtained from the definition by the lower equation of Eq. (3.8) as follows for Table 4.2-1 and Table 4.2-2;

For the models of Table 4.2-1;

$$\text{Type 1:} \quad L^* = 2\pi \frac{S_{ff}(0)}{\sigma_{ff}^2} = 2\pi \frac{b}{2} = \pi b$$

$$\text{Type 2:} \quad L_F^* = \frac{1}{\pi\sqrt{2}}\left(2\pi \frac{\sigma_{FF}}{\sigma_{ff}}\right) = \frac{1}{\pi\sqrt{2}}\left(2\pi \frac{b}{\sqrt{2}}\right) = b$$

$$\text{Type 3:} \quad L_F^* = \frac{1}{\pi\sqrt{2}}\left(2\pi \frac{\sigma_{FF}}{\sigma_{ff}}\right) = \frac{1}{\pi\sqrt{2}}\left(2\pi \frac{b}{\sqrt{12}}\right) = \frac{b}{\sqrt{6}}$$

$$\text{Type 4:} \quad L_F^* = \frac{1}{\pi\sqrt{2}}\left(2\pi \frac{\sigma_{FF}}{\sigma_{ff}}\right) = \frac{1}{\pi\sqrt{2}}\left(2\pi \frac{b}{\sqrt{30}}\right) = \frac{b}{\sqrt{15}}$$

$$\text{Type 5:} \quad L_F^* = \frac{1}{\pi\sqrt{2}}\left(2\pi \frac{\sigma_{FF}}{\sigma_{ff}}\right) = \frac{1}{\pi\sqrt{2}}\left(2\pi \frac{b}{\sqrt{56}}\right) = \frac{b}{\sqrt{28}}$$

$$\text{Type 6:} \quad L_F^* = \frac{1}{\pi\sqrt{2}}\left(2\pi \frac{\sigma_{FF}}{\sigma_{ff}}\right) = \frac{1}{\pi\sqrt{2}}\left(2\pi \frac{b}{\sqrt{90}}\right) = \frac{b}{\sqrt{45}}$$

For the models of Table 4.2-2;

$$\text{Type 1:} \quad L^* = 2\pi \frac{S_{ff}(0)}{\sigma_{ff}^2} = 2\pi \frac{b}{2\sqrt{\pi}} = b\sqrt{\pi}$$

$$\text{Type 2:} \quad L_F^* = \frac{1}{\pi\sqrt{2}}\left(2\pi \frac{\sigma_{FF}}{\sigma_{ff}}\right) = \frac{1}{\pi\sqrt{2}}\left(2\pi \frac{b}{\sqrt{2}}\right) = b$$

$$\text{Type 3:} \quad L_F^* = \frac{1}{\pi\sqrt{2}}\left(2\pi \frac{\sigma_{FF}}{\sigma_{ff}}\right) = \frac{1}{\pi\sqrt{2}}\left(2\pi \frac{b}{\sqrt{6}}\right) = \frac{b}{\sqrt{3}}$$

Type 4: $L_F^* = \dfrac{1}{\pi\sqrt{2}}\left(2\pi\dfrac{\sigma_{FF}}{\sigma_{ff}}\right) = \dfrac{1}{\pi\sqrt{2}}\left(2\pi\dfrac{b}{\sqrt{10}}\right) = \dfrac{b}{\sqrt{5}}$

Type 5: $L_F^* = \dfrac{1}{\pi\sqrt{2}}\left(2\pi\dfrac{\sigma_{FF}}{\sigma_{ff}}\right) = \dfrac{1}{\pi\sqrt{2}}\left(2\pi\dfrac{b}{\sqrt{14}}\right) = \dfrac{b}{\sqrt{7}}$

Type 6: $L_F^* = \dfrac{1}{\pi\sqrt{2}}\left(2\pi\dfrac{\sigma_{FF}}{\sigma_{ff}}\right) = \dfrac{1}{\pi\sqrt{2}}\left(2\pi\dfrac{b}{\sqrt{18}}\right) = \dfrac{b}{\sqrt{9}}$

Two examples of exact behavior of variance ratio $\sigma_D^2 / \sigma_{ff}^2$ using Type 2 and Type 3 of Table 4.2-1 and Table 4.2-2 are shown as follows,

In the case of Type 2 of Table 4.2-1, the power spectral density function of $f(x)$ and the correlation function of $F(x)$ are given by,

$$
\begin{aligned}
S_{ff}(\kappa) &= \frac{\sigma_{ff}^2}{4} b^3 \kappa^2 e^{-b|\kappa|} \\
R_{FF}(\xi) &= \int_{-\infty}^{\infty} \frac{S_{ff}(\kappa)}{\kappa^2} \cos\kappa\xi d\kappa = \frac{\sigma_{ff}^2}{2} \frac{b^4}{b^2+\xi^2}
\end{aligned}
\tag{4.2-1a}
$$

Substituting the above correlation function of Type 2 of Table 4.2-1 to Eq. (2.14b), $\sigma_D^2 / \sigma_{ff}^2$ is given such as,

$$
\begin{aligned}
\frac{\sigma_D^2}{\sigma_{ff}^2} &= \frac{2}{D^2\sigma_{ff}^2}\left(R_{FF}(0) - R_{FF}(D)\right) \\
&= \left(\frac{b}{D}\right)^2\left(1 - \frac{1}{1+(D/b)^2}\right) \\
&= \left(\frac{L_F^*}{D}\right)^2\left(1 - \frac{1}{1+(D/L_F^*)^2}\right), \quad L_F^* = b
\end{aligned}
\tag{4.2-1b}
$$

The above equation gives for the two extreme cases,

$$\frac{\sigma_D^2}{\sigma_{ff}^2} = \begin{cases} 1 & D \to 0 \\ \left(\dfrac{L_F^*}{D}\right)^2 & D \to \infty \end{cases} \tag{4.2-1c}$$

In the case of Type 3 of Table 4.2-2, the power spectral density function of $f(x)$ and the correlation function of $F(x)$ are given by,

$$\begin{aligned} S_{ff}(\kappa) &= \frac{1}{2}\frac{\sigma_{ff}^2}{12\sqrt{\pi}} b^5 \kappa^4 e^{-(b\kappa/2)^2} \\ R_{FF}(\xi) &= \int_{-\infty}^{\infty} \frac{S_{ff}(\kappa)}{\kappa^2} \cos\kappa\xi d\kappa = \left(\frac{\sigma_{ff}^2 b^2}{6}\right)\left(1 - 2\left(\frac{\xi}{b}\right)^2\right) e^{-(\xi/b)^2} \end{aligned} \tag{4.2-2a}$$

Substituting the above correlation function of Type 3 of Table 4.2-2 to Eq. (2.14b), $\sigma_D^2 / \sigma_{ff}^2$ is given such as,

$$\begin{aligned} \frac{\sigma_D^2}{\sigma_{ff}^2} &= \frac{2}{D^2\sigma_{ff}^2}\left(R_{FF}(0) - R_{FF}(D)\right) \\ &= \frac{1}{3}\left(\frac{b}{D}\right)^2 \left(1 - \left(1 - 2(D/b)^2\right)\right) e^{-(D/b)^2} \\ &= \left(\frac{L_F^*}{D}\right)^2 \left(1 - \left(1 - \frac{2}{3}(D/L_F^*)^2\right)\right) e^{-\frac{1}{3}(D/L_F^*)^2}, \quad L_F^* = \frac{b}{\sqrt{3}} \end{aligned} \tag{4.2-2b}$$

The above equation gives for the two extreme cases using Eq. (4.1-2d),

$$\frac{\sigma_D^2}{\sigma_{ff}^2} = \begin{cases} 1 & D \to 0 \\ \left(\dfrac{L_F^*}{D}\right)^2 & D \to \infty \end{cases} \tag{4.2-2c}$$

Table 4.2-3 summarize the correlation scales for various correlation functions and power spectral density functions in Table 4.1-1 (Case I $S_{ff}(0) \neq 0$), Table 4.2-1 (Case II $S_{ff}(0) = 0$) and Table 4.2-2 (Case II $S_{ff}(0) = 0$).

Table 4.2-3 Scale of Correlation of $f(x)$ for Correlation Functions in Table 4.1-1, Table 4.2-1 and Table 4.2-2

Type of Table4.1-1	L^*	L_F^*	Type of Table4.2-1	L^*	L_F^*	Type of Table4.2-2	L^*	L_F^*
1 [I]	b	0	1 [I]	πb	0	1 [I]	$\sqrt{\pi}b$	0
2 [I]	$2b$	0	2 [II]	0	b	2 [II]	0	b
3 [I]	$4b$	0	3 [II]	0	$\frac{b}{\sqrt{6}}$	3 [II]	0	$\frac{b}{\sqrt{3}}$
4 [I]	$\frac{16}{3}b$	0	4 [II]	0	$\frac{b}{\sqrt{15}}$	4 [II]	0	$\frac{b}{\sqrt{5}}$
5 [I]	$\sqrt{\pi}b$	0	5 [II]	0	$\frac{b}{\sqrt{28}}$	5 [II]	0	$\frac{b}{\sqrt{7}}$
6 [I]	πb	0	6 [II]	0	$\frac{b}{\sqrt{45}}$	6[II]	0	$\frac{b}{\sqrt{9}}$

[I] and [II] mean Case I and Case II. Zero of scale of correlation means that correlation scale is not defined.

5. GRAPHICAL REPRESENTATION

A graphical representation of Eqs. (3.9a) and (3.9b) as shown in Fig. 5.1 may be more useful for estimating the correlation scale L^* or L_F^* from a set of observed data. Figure 5.1 is constructed in the following way.

Plotting σ_D / σ_{ff} for the two limiting cases indicated in Eqs. (3.9a) and (3.9b) as a function of D / L^* or D / L_F^* in log-log scale, we can obtain a diagram (heavy solid lines) as shown in Fig. 5.1: From Eq. (3.9a),

$$\log\frac{\sigma_D}{\sigma_{ff}} = \begin{cases} \log 1 & D \le L^* \\ -\dfrac{1}{2}\log\dfrac{D}{L^*} & D > L^* \end{cases} \tag{5.1a}$$

Also, from Eq. (2.15), the deviation σ_{F_D} of the difference process $F_D(x)$ is expressed and its logarithmic expression as,

$$\begin{aligned} \frac{\sigma_D}{\sigma_{ff}} &= \frac{1}{D}\frac{\sigma_{F_D}}{\sigma_{ff}} = \frac{L^*}{D}\frac{\sigma_{F_D}}{\sigma_{ff}L^*} \\ \log\frac{\sigma_D}{\sigma_{ff}} &= -\log\frac{D}{L^*} + \log\frac{\sigma_{F_D}}{\sigma_{ff}L^*} \end{aligned} \tag{5.1b}$$

From Eq. (3.9b),

$$\log\frac{\sigma_D}{\sigma_{ff}} = \begin{cases} \log 1 & D \le L_F^* \\ -\log\dfrac{D}{L_F^*} & D > L_F^* \end{cases} \tag{5.2a}$$

By using $L_F^* = \sqrt{\pi}\sigma_{FF} / \sigma_{ff}$ (see Eq. (3.8)), the deviation σ_{F_D} of the difference process $F_D(x)$ is expressed and its logarithmic expression as,

$$\frac{\sigma_D}{\sigma_{ff}} = \frac{1}{D}\frac{\sigma_{F_D}}{\sigma_{ff}} = \frac{L_F^*}{D}\frac{\sigma_{F_D}}{\sigma_{ff}L_F^*} = \frac{L_F^*}{D}\frac{\sigma_{F_D}}{\sigma_{FF}\sqrt{2}}$$

$$\log\frac{\sigma_D}{\sigma_{ff}} = -\log\frac{D}{L_F^*} + \log\frac{\sigma_{F_D}}{\sigma_{FF}\sqrt{2}} \tag{5.2b}$$

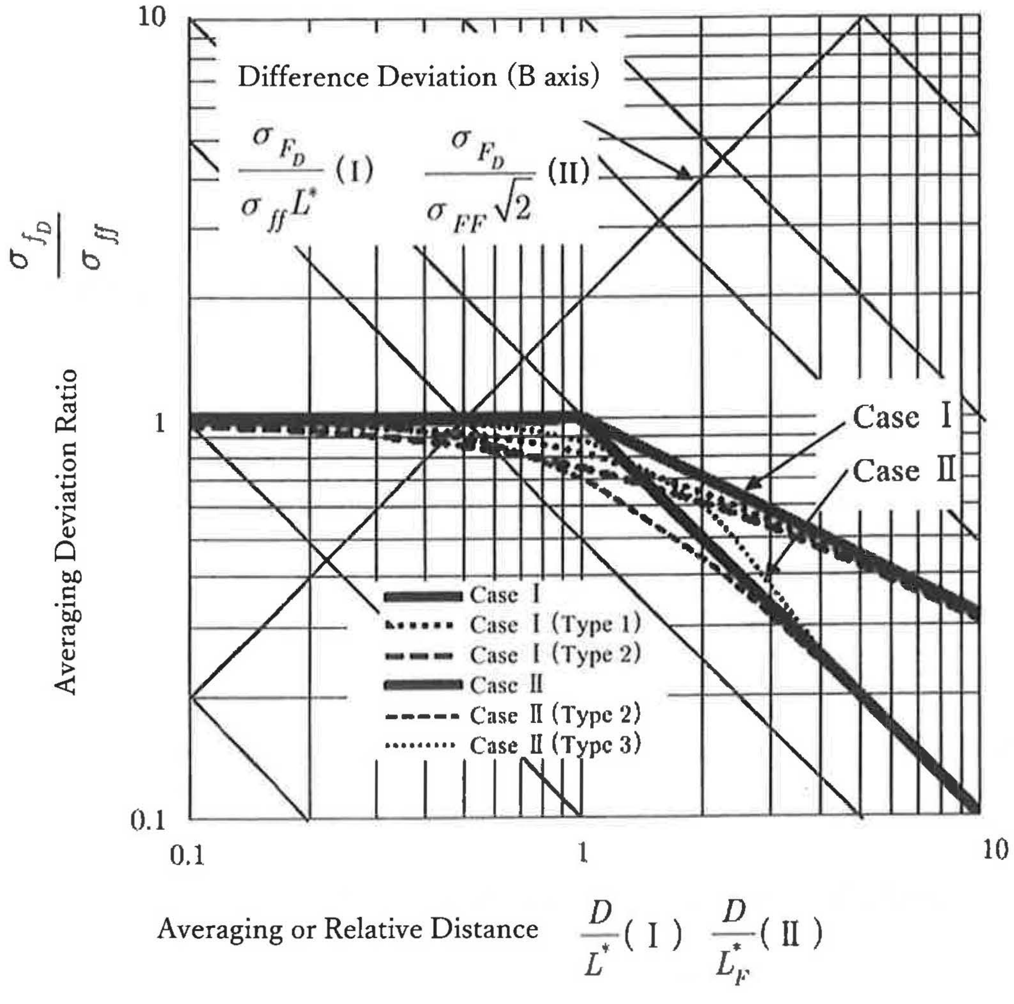


Fig. 5.1 $\sigma_D - D$ Diagram

Due to Eqs. (5.1a) and (5.2b), along a straight line making a 45 degree with the horizontal axis $\log D / L^*$ or $\log D / L_F^*$ in Fig. 5.1, the value of σ_{F_D} is constant and hence, the axis of B of $\log \sigma_{F_D} / \sigma_{ff} L^*$ or $\log \sigma_{F_D} / \sigma_{FF}\sqrt{2}$ can be constructed.

A set of eight samples of the exact $\sigma_D - D$ relationships are also plotted in Fig. 5.1 (dashed curves) using the particular forms of the correlation function for the original process $f(x)$ which are designated as Types 2 and 3, respectively in Tables 4.2-1 and 4.2-2 for Case II, and as Types 1, 2, 5 and 6 in Table 4.1-1 for Case I.

It can be observed from Fig. 5.1 that all these curves asymptotically approach the heavy solid lines in the ranges where $D \to 0$ and $D \to \infty$, and that in the intermediate range of D, the heavy solid lines tend to represent the average or upper bound trend of all the dashed curves. Using a diagram as shown in Fig. 5.1, the correlation scale L^* or L_F^* can be determined from the length D at the intersection of heavy slid lines in Fig. 5.1 if such a diagram is constructed as a function of D using the variances (or deviation) estimated from observed data (see Chapter 6) .

It should be noted here that all the pairs of correlation functions and power spectral density functions indicated in Tables 4.1-1, 4.2-1 and 4.2-2 except Types 1 and 2 in Tables 4.2-1 and 4.2-2 have at least first order derivative processes. Hence, they are used as the correlation function or power spectral density function of the homogeneous process not only $F(x)$ but also $f(x)$. The correlation scale L_F^* are obtained by interpreting the correlation functions as those of the integral process $F(x)$.

As shown in this example (Table 4.2-3), we must classify the original process $f(x)$ into two cases (Case I and Case II) to obtain a physically meaningful correlation scale for the stochastic process $f(x)$. In this sense, a real phenomenon may be modeled by the Case I process, the case II process or the combined process of the Case I and Case II process. By appropriately combined process of the Case I process and Case II process, we can construct a more sophisticated homogeneous stochastic process model (two dimensional stochastic fields, Chapter 7) which may be able to more accurately represent real phenomena.

6. PRACTICAL ESTIMATION PROCEDURE AND NUMERICAL EXAMPLE

In this chapter, to illustrate the most important aim of this study that the correlation statistics (correlation scale) is estimated from observed data without using the correlation function (Chapter 1), we describe a practical procedure for estimating the correlation scale from finite length observed data with numerical example.

6.1 Practical Estimation Procedure

The procedure is as follows (see Fig. 6.1):

(1) From a set of observed data, estimate the mean value $\tilde{m}$ and variance $\tilde{\sigma}_{ff}^2$,

(2) Obtain a set of averaging process data for several large values of D and calculate the variance $\tilde{\sigma}_D^2$ from them,

(3) Plot a set of averaging deviation ratio $\tilde{\sigma}_D \,/\, \tilde{\sigma}_{ff}$ on a log-log scaled graph as a function of D,

(4) If the estimated deviation ratio $\tilde{\sigma}_D \,/\, \tilde{\sigma}_{ff}$ follows straight line I, determine the correlation scale L^* from the length D of the intersection between horizontal line and straight line I. If the ratio $\tilde{\sigma}_D \,/\, \tilde{\sigma}_{ff}$ follows

straight line II, determine the correlation scale L_F^* from the length D of the intersection between the horizontal line and straight line II.

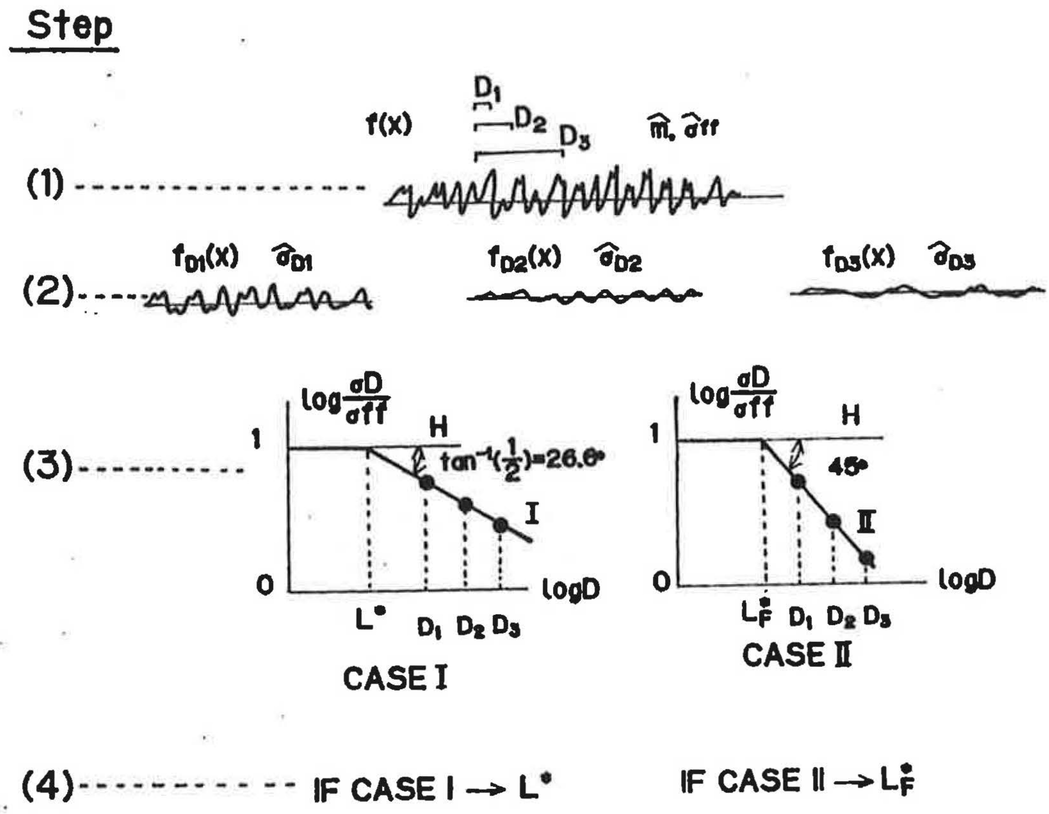


Fig. 6.1 Practical Estimation Procedure of the Correlation Scale

6.2 Numerical Example

A numerical example is based on digitally simulated stochastic data using the following equation (Shinozuka and Yang (1972)):

$$f(x) = \sqrt{2}\sum_{n=1}^{N}\sqrt{2S_{ff}(\kappa_n)d\kappa}\cos\left(\kappa_n x + \theta_n\right) \tag{6.2-1}$$

where θ_n is the random phase angle uniformly distributed between 0 and 2π,

$d\kappa = \kappa_u / N$, $\kappa_n = nd\kappa$ and κ_u is upper cut-off wavenumber where $S_{ff}(\kappa_u)$ is approximately zero.

For numerical example, the following data are used:

Type 1 in Table 4.2-2 (Case I) with $b = 31.636\,\mathrm{m}$,

$\kappa_u = 1\,\mathrm{rad/m}$ and $\sigma_{ff}^2 = 1\,\mathrm{m}$.

By using these data, a sample function of $f(x)$ is shown for the distance of 2,000 m in Fig. 6.2-1.

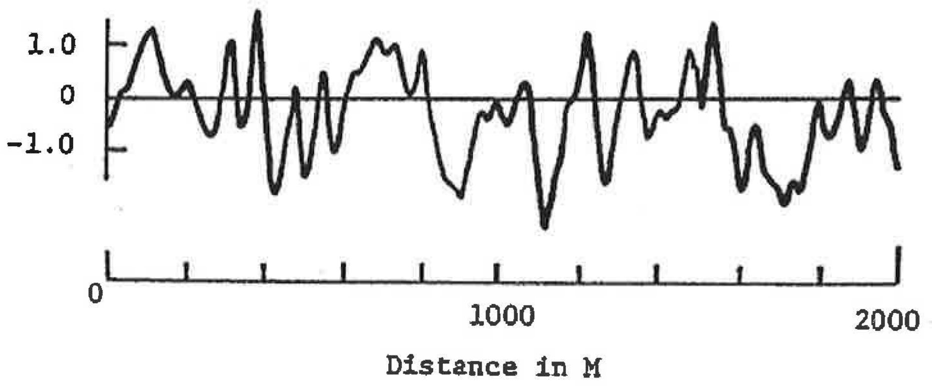


Fig. 6.2-1 Sample Function of $f(x)$

For D=200, 300, 400, 500 m, the variances $\tilde{\sigma}_D^2$ are calculated and the resulting deviation ratios $\tilde{\sigma}_D / \tilde{\sigma}_{ff}$ are plotted in Fig. 6.2-2. From Fig. 6.2-2, L^* is estimated to be about 56 m. In fact, in this numerical example, the true correlation scale $L^* = b\sqrt{\pi} = 56.07\,\mathrm{m}$.

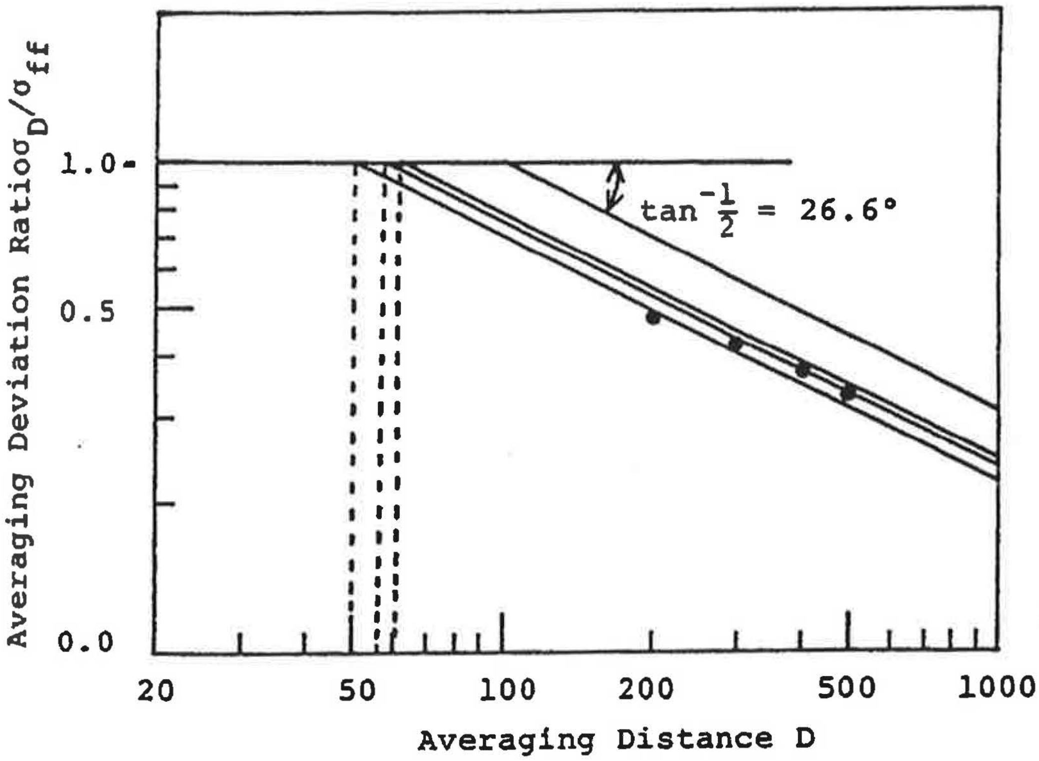


Fig. 6.2-2 Numerical Example for Graphical Estimation of the Scale of Correlation

6.3 Statistical Assessments of Correlation Scale Estimates

To establish the quality of the estimator, we will use two principal factors in this study: "unbiased" and "consistent", that is,

$$E\left[\tilde{\phi}\right] = \phi \qquad \text{unbiased} \qquad (6.3\text{-}1a)$$

and

$$\lim E\left[\left(\tilde{\phi} - \phi\right)^2\right] = 0 \qquad \text{consistent} \qquad (6.3\text{-}1b)$$

where $E[\tilde{\phi}]$ is expectation of $\tilde{\phi}$ being an estimator for the parameter ϕ. The analysis procedures that follow are basically based on those by Bendat and Piersol (1971).

(1) Mean Values

Consider the sample record $\tilde{f}(x)$ from a homogeneous stochastic process over a finite length D_0. The mean value can be estimated by

$$\tilde{m} = \frac{1}{D_0}\int_0^{D_0} \tilde{f}(x)dx \tag{6.3-2a}$$

and true value is

$$m = E\left[\tilde{f}(x)\right] \tag{6.3-2b}$$

Then, the expected value of $\tilde{m}$ is,

$$E\left[\tilde{m}\right] = \frac{1}{D_0}\int_0^{D_0} E\left[\tilde{f}(x)\right]dx = m \tag{6.3-3a}$$

Hence, $\tilde{m}$ is an unbiased estimate of m in accordance with Eq. (6.3-1a).

The variance of $\tilde{m}$ is expressed as,

$$\mathrm{Var}\left[\tilde{m}\right] = E\left[(\tilde{m} - m)^2\right] = E\left[\tilde{m}^2\right] - m^2 \tag{6.3-3b}$$

Introducing the covariance function $C_{\tilde{f}\tilde{f}}(\xi)$ of $\tilde{f}(x)$ as,

$$\begin{aligned} C_{\tilde{f}\tilde{f}}(\xi) &= E\left[\left(\tilde{f}(x+\xi) - m\right)\left(\tilde{f}(x) - m\right)\right] \\ &= R_{\tilde{f}\tilde{f}}(\xi) - m^2 \\ R_{\tilde{f}\tilde{f}}(\xi) &= E\left[\tilde{f}(x+\xi)\tilde{f}(x)\right] \end{aligned} \tag{6.3-3c}$$

Then, the variance of $\tilde{m}$ is written in terms of the covariance function as follows,

$$\mathrm{Var}\left[\tilde{m}\right] = \frac{1}{D_0^2}\int_0^{D_0}\int_0^{D_0}\left(E\left[\tilde{f}(\xi_1)\tilde{f}(\xi_2)\right] - m^2\right)d\xi_1 d\xi_2$$

$$= \frac{1}{D_0^2}\int_0^{D_0}\int_0^{D_0} C_{\tilde{f}\tilde{f}}(\xi_1 - \xi_2)d\xi_1 d\xi_2 \tag{6.3-3d}$$

$$= \frac{1}{D_0}\int_{-D_0}^{D_0}\left(1 - \frac{|\xi|}{D_0}\right)C_{\tilde{f}\tilde{f}}(\xi)d\xi$$

In deriving the above last equation, we change the region of integration from (ξ_1, ξ_2) to (ξ, ξ_2) by $\xi = \xi_1 - \xi_2$, $d\xi = d\xi_1$. Then, the integration yields,

$$\begin{aligned}\int_0^{D_0}\int_0^{D_0} C_{\tilde{f}\tilde{f}}(\xi_1 - \xi_2)d\xi_1 d\xi_2 &= \int_{-D_0}^{0}\int_{-\xi}^{D_0} C_{\tilde{f}\tilde{f}}(\xi)d\xi d\xi_2 + \\ &\quad \int_0^{D_0}\int_0^{D_0-\xi} C_{\tilde{f}\tilde{f}}(\xi)d\xi d\xi_2 \\ &= \int_{-D_0}^{0}\left(D_0 + \xi\right)C_{\tilde{f}\tilde{f}}(\xi)d\xi + \\ &\quad \int_0^{D_0}\left(D_0 - \xi\right)C_{\tilde{f}\tilde{f}}(\xi)d\xi \\ &= D_0\int_{-D_0}^{D_0}\left(1 - \frac{|\xi|}{D_0}\right)C_{\tilde{f}\tilde{f}}(\xi)d\xi\end{aligned} \tag{6.3-3e}$$

For $D_0 \to \infty$, the above equation is written as,

$$\mathrm{Var}\left[\tilde{m}\right] = \frac{1}{D_0}\int_{-\infty}^{\infty} C_{\tilde{f}\tilde{f}}(\xi)d\xi \tag{6.3-3f}$$

Hence, $\mathrm{Var}\left[\tilde{m}\right]$ approaches zero as $D_0 \to \infty$, indicating that $\tilde{m}$ is a consistent estimate of the mean value m.

(2) Variance

Since the mean value can be estimated unbiasedly and consistently by Eq. (6.3-2a), we consider the stochastic process $f(x)$ with zero mean and variance σ_{ff}^2. The variance of $\tilde{f}(x)$ and the true variance σ_{ff}^2 may be estimated by,

$$\begin{aligned} \tilde{\sigma}_{ff}^2 &= \frac{1}{D_0}\int_0^{D_0} \tilde{f}^2(x)dx \\ \sigma_{ff}^2 &= E\left[\tilde{f}^2(x)\right] \end{aligned} \tag{6.3-4}$$

The expected value of the estimate $\tilde{\sigma}_{ff}^2$ is,

$$E\left[\tilde{\sigma}_{ff}^2\right] = \frac{1}{D_0}\int_0^{D_0} E\left[\tilde{f}^2(x)\right]dx = \sigma_{ff}^2 \tag{6.3-5}$$

Hence, $\tilde{\sigma}_{ff}^2$ is an unbiased estimate of σ_{ff}^2 in accordance with Eq. (6.3-1a).

The variance of $\tilde{\sigma}_{ff}^2$ is expressed as,

$$\begin{aligned} \mathrm{Var}\left[\tilde{\sigma}_{ff}^2\right] &= E\left[\left(\tilde{\sigma}_{ff}^2 - \sigma_{ff}^2\right)^2\right] = E\left[\tilde{\sigma}_{ff}^4\right] - \sigma_{ff}^4 \\ &= \frac{1}{D_0^2}\int_0^{D_0}\int_0^{D_0}\left(E\left[\tilde{f}^2(\xi_1)\tilde{f}^2(\xi_2)\right] - \sigma_{ff}^4\right)d\xi_1 d\xi_2 \end{aligned} \tag{6.3-6a}$$

Assume now that $f(x)$ is a Gaussian stochastic process. Then the expected value in Eq. (6.3-6a) can be expressed in terms of second order statistics such as (Bendat and Piersol (1971)),

$$E\left[\tilde{f}^2(\xi_1)\tilde{f}^2(\xi_2)\right] = 2R_{\tilde{f}\tilde{f}}^2(\xi_1 - \xi_2) + \sigma_{ff}^4 \tag{6.3-6b}$$

Substitution of Eq. (6.3-6b) into Eq. (6.3-6a) yields,

$$\begin{aligned}\mathrm{Var}\left[\tilde{\sigma}_{ff}^2\right] &= \frac{2}{D_0^2}\int_0^{D_0}\int_0^{D_0} R_{\tilde{f}\tilde{f}}^2(\xi_1-\xi_2)d\xi_1 d\xi_2 \\ &= \frac{2}{D_0}\int_{-D_0}^{D_0}\left(1-\frac{|\xi|}{D_0}\right)R_{\tilde{f}\tilde{f}}^2(\xi)d\xi\end{aligned} \tag{6.3-6c}$$

For large $D_0 \to \infty$, the variance becomes,

$$\mathrm{Var}\left[\tilde{\sigma}_{ff}^2\right] = \frac{2}{D_0}\int_{-\infty}^{\infty} R_{\tilde{f}\tilde{f}}^2(\xi)d\xi \tag{6.3-6d}$$

Thus the $\tilde{\sigma}_{ff}^2$ estimated by Eq. (6.3-4) is a consistent estimate of σ_{ff}^2 because $\mathrm{Var}\left[\tilde{\sigma}_{ff}^2\right]$ approaches zero as $D_0 \to \infty$ assuming a finite value of the integral.

(3) Correlation Scale

For the reason that the mean value and variance estimators $\tilde{m}$ and $\tilde{\sigma}_{ff}^2$ of $\tilde{f}(x)$ are unbiased and consistent estimates of $f(x)$ as shown in prior items (1) and (2), we restrict our attention here to processes with zero mean and unit variance. Then the correlation scales L^* and L_F^* may be estimated from Eqs. (3.9a) and (3.9b) as,

$$\tilde{L}^* = D\tilde{\sigma}_D^2 \tag{6.3-7a}$$

and

$$\tilde{L}_F^* = D\tilde{\sigma}_D \tag{6.3-7b}$$

where $\tilde{\sigma}_D^2$ is a variance estimate of the averaging process $\tilde{f}_D(x)$ of $\tilde{f}(x)$. It is estimated by,

$$\tilde{\sigma}_D^2 = \frac{1}{D_0 - D} \int_{D_0 - D/2}^{D_0 + D/2} \tilde{f}_D^2(x)dx \simeq \frac{1}{D_0} \int_0^{D_0} \tilde{f}_D^2(x)dx \qquad \text{(6.3-8)}$$

In Eqs. (6.3-7) and (6.3-8), $\tilde{L}^*$ or $\tilde{L}_F^* \ll D \ll D_0$ is assumed.

The expected value of $\tilde{\sigma}_D^2$ is,

$$E\left[\tilde{\sigma}_D^2\right] = \frac{1}{D_0} \int_0^{D_0} E\left[\tilde{f}_D^2(x)\right]dx = \sigma_D^2 \qquad \text{(6.3-9a)}$$

where σ_D^2 is the true variance of the averaging process $f_D(x)$. Hence, the expectation of $\tilde{L}^*$ and $\tilde{L}_F^*$ are given by,

$$E\left[\tilde{L}^*\right] = DE\left[\tilde{\sigma}_D^2\right] = D\sigma_D^2 = L^* \qquad \text{(6.3-9b)}$$

and

$$E\left[\tilde{L}_F^*\right] = DE\left[\tilde{\sigma}_D\right] = D\sigma_D = L_F^* \qquad \text{(6.3-9c)}$$

Therefore, $\tilde{L}^*$ and $\tilde{L}_F^*$ are the unbiased estimates of L^* and L_F^*, respectively.

Next, the variance of $\tilde{\sigma}_D^2$ is given by,

$$\text{Var}\left[\tilde{\sigma}_D^2\right] = E\left[\left(\tilde{\sigma}_D^2 - \sigma_D^2\right)^2\right] \qquad \text{(6.3-10)}$$

Then the variances of $\tilde{L}^*$ and $\tilde{L}_F^*$ are expressed such that,

$$\text{Var}\left[\tilde{L}^*\right] = D^2\text{Var}\left[\tilde{\sigma}_D^2\right] \qquad \text{(6.3-11a)}$$

and

$$\text{Var}\left[\tilde{L}_F^*\right] = D^2\sqrt{\text{Var}\left[\tilde{\sigma}_D^2\right]} \qquad \text{(6.3-11b)}$$

Similar to Eq. (6.3-6d), Eq. (6.3-10) can be written for large $D_0 \to \infty$,

$$\mathrm{Var}\left[\tilde{\sigma}_D^2\right] = \frac{2}{D_0} \int_{-\infty}^{\infty} R_{f_D}^2(\xi) d\xi \tag{6.3-12a}$$

where $R_{f_D}(\xi)$ is the correlation function of $f_D(x)$ given by

$$\begin{aligned} R_{f_D}(\xi) &= E\left[f_D(x+\xi) f_D(x)\right] \\ &= \frac{1}{D^2} \int_{x+\xi-D/2}^{x+\xi+D/2} \int_{x-D/2}^{x-D/2} E\left[f(y)f(z)\right] dy dz \\ &= \frac{1}{D^2} \int_0^D \int_0^D R_{ff}(\xi + \xi_1 - \xi_2) d\xi_1 d\xi_2 \\ &= \frac{1}{D} \int_{-D}^{D} \left(1 - \frac{|\xi_0|}{D}\right) R_{ff}(\xi + \xi_0) d\xi_0 \end{aligned} \tag{6.3-12b}$$

If the infinite integral process $F(x)$ of $f(x)$ is homogeneous, $R_{f_D}(\xi)$ is given by

$$\begin{aligned} R_{f_D}(\xi) &= \frac{1}{D^2} E\left[\begin{matrix} \left(F\left(x+\xi+\frac{D}{2}\right) - F\left(x+\xi-\frac{D}{2}\right)\right) \times \\ \left(F\left(x+\frac{D}{2}\right) - F\left(x-\frac{D}{2}\right)\right) \end{matrix}\right] \\ &= \frac{1}{D^2}\left(2R_{FF}(\xi) - R_{FF}(\xi+D) - R_{FF}(\xi-D)\right) \end{aligned} \tag{6.3-12c}$$

where $R_{FF}(\xi)$ is the correlation function of $F(x)$. For $\tilde{L}^*$ or $\tilde{L}_F^* \ll D \ll D_0$, Eqs. (6.3-12b) and (6.3-12c) may be written as,

$$R_{f_D}(\xi) = \frac{1}{D} \int_{-\infty}^{\infty} R_{ff}(\xi + \xi_0) d\xi_0 \tag{6.3-13a}$$

and

$$R_{f_D}(\xi) = \frac{2}{D^2} R_{FF}(\xi) \tag{6.3-13b}$$

From Eqs. (6.3-12a), (6.3-13a) and (6.3-13b), the variance of $\tilde{\sigma}_D^2$ is given by

$$\mathrm{Var}\left[\tilde{\sigma}_D^2\right] = \frac{2}{D_0 D^2} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} R_{ff}(\xi + \xi_0) d\xi_0 \right)^2 d\xi \quad \text{for } S_{ff}(0) \neq 0 \qquad (6.3\text{-}14a)$$

and

$$\mathrm{Var}\left[\tilde{\sigma}_D^2\right] = \frac{8}{D_0 D^4} \int_{-\infty}^{\infty} R_{FF}^2(\xi) d\xi \qquad \text{for } S_{ff}(0) = 0 \qquad (6.3\text{-}14b)$$

Hence the variances of the correlation scale estimates are from Eqs. (6.3-11a) and (6.3-11b),

$$\mathrm{Var}\left[\tilde{L}^*\right] = \frac{2}{D_0} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} R_{ff}(\xi + \xi_0) d\xi_0 \right)^2 d\xi \quad \text{for } S_{ff}(0) \neq 0 \qquad (6.3\text{-}15a)$$

and

$$\mathrm{Var}\left[\tilde{L}_F^*\right] = \sqrt{\frac{8}{D_0} \int_{-\infty}^{\infty} R_{FF}^2(\xi) d\xi} \qquad \text{for } S_{ff}(0) = 0 \qquad (6.3\text{-}15b)$$

Thus, $\tilde{L}^*$ and $\tilde{L}_F^*$ given by Eqs. (6.3-7a) and (6.3-7b) are consistent estimates of L^* and L_F^* since $\mathrm{Var}\left[\tilde{L}^*\right]$ and $\mathrm{Var}\left[\tilde{L}_F^*\right]$ approach zero as $D_0 \to \infty$. Equation (6.3-15a) is identical with that derived by Vanmarke (1983). It should again be noted that Eqs. (6.3-15a) and (6.3-15b) are derived from the assumption that $f(x)$ is homogeneous Gaussian processes with zero mean and unit variance. Hence the correlation function $R_{ff}(\xi)$ in Eq. (6.3-15a) is normalized with $R_{ff}(0) = \sigma_{ff}^2 = 1$.

7. CORRELATION SCALES OF TWO DIMENSIONAL FIELDS

In this chapter, we briefly discuss the correlation scales of two dimensional stochastic fields by extending the procedures developed in the previous chapters. And also, numerical examples of two dimensional stochastic fields and correlation scales using the 3 separable two dimensional power spectral density functions in order to visually illustrate the correlation scales and patterns of two dimensional stochastic fields.

7.1 Variance of Averaging Process

For the original homogeneous stochastic field $f(x,y)$ with zero mean and variance σ_{ff}^2, the averaging field $f_A(x,y)$ may be defined as,

$$f_A(x,y) = \frac{1}{A} \int_{x-\frac{D_x}{2}}^{x+\frac{D_x}{2}} \int_{y-\frac{D_y}{2}}^{y+\frac{D_y}{2}} f(u,v)dudv \qquad (7.1\text{-}1)$$

where $A = D_x D_y$ and D_x, D_y are the averaging distances of the x and y coordinates, respectively.

Introducing the following indefinite integrals,

$$F_x(x,y)=\int f(x,y)dx \quad \text{or} \quad \frac{\partial F_x(x,y)}{\partial x}=f(x,y)$$

$$F_y(x,y)=\int f(x,y)dy \quad \text{or} \quad \frac{\partial F_y(x,y)}{\partial y}=f(x,y) \tag{7.1-2}$$

$$F(x,y)=\int f(x,y)dx \quad \text{or} \quad \frac{\partial^2 F(x,y)}{\partial x \partial y}=f(x,y)$$

If they are homogeneous, the power spectral density functions of $F_x(x,y)$, $F_y(x,y)$ and $F(x,y)$ are given by,

$$S_{F_xF_x}(\kappa_x,\kappa_y)=\frac{S_{ff}(\kappa_x,\kappa_y)}{\kappa_x^2}$$

$$S_{F_yF_y}(\kappa_x,\kappa_y)=\frac{S_{ff}(\kappa_x,\kappa_y)}{\kappa_y^2} \tag{7.1-2}$$

$$S_{FF}(\kappa_x,\kappa_y)=\frac{S_{ff}(\kappa_x,\kappa_y)}{\kappa_x^2\kappa_y^2}$$

where $S_{ff}(\kappa_x,\kappa_y)$ is the power spectral density function of $f(x,y)$. As is well known, $S_{ff}(\kappa_x,\kappa_y)$ is related to the correlation function $R_{ff}(\xi_x,\xi_y)$ through the Wiener Kintchine transform pair,

$$S_{ff}(\kappa_x,\kappa_y)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{ff}(\xi_x,\xi_y)e^{-i(\kappa_x\xi_x+\kappa_y\xi_y)}d\xi_x d\xi_y$$

$$R_{ff}(\xi_x,\xi_y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{ff}(\kappa_x,\kappa_y)e^{i(\kappa_x\xi_x+\kappa_y\xi_y)}d\kappa_x d\kappa_y \tag{7.1-3a}$$

where κ_x,κ_y and ξ_x,ξ_y are the wavenumbers and the separation distances of the x and y coordinates, respectively. By taking into account the relations,

$$
\begin{aligned}
R_{ff}(\xi_x, \xi_y) &= R_{ff}(-\xi_x, \xi_y) \\
R_{ff}(\xi_x, -\xi_y) &= R_{ff}(-\xi_x, \xi_y)
\end{aligned}
\tag{7.1-3b}
$$

Eq. (7.1-3a) is also expressed as,

$$
\begin{aligned}
S_{ff}(\kappa_x, \kappa_y) &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{ff}(\xi_x, \xi_y) \cos\left(\kappa_x \xi_x + \kappa_y \xi_y\right) d\xi_x d\xi_y \\
R_{ff}(\xi_x, \xi_y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{ff}(\kappa_x, \kappa_y) \cos\left(\kappa_x \xi_x + \kappa_y \xi_y\right) d\kappa_x d\kappa_y
\end{aligned}
\tag{7.1-3c}
$$

Similar to the discussions in chapter 2, the integral processes $F_x(x, y)$, $F_y(x, y)$ and $F(x, y)$ are not always homogeneous. The conditions of homogeneity of the integral processes depend on the behavior of the power spectral density function $S_{ff}(\kappa_x, \kappa_y)$ of the original process $f(x, y)$ at the origin $\kappa_x = \kappa_y = 0$. Using asymptotic expansion of $\cos(\kappa_x \xi_x + \kappa_y \xi_y)$, $S_{ff}(\kappa_x, \kappa_y)$ can be expressed as,

$$
\begin{aligned}
S_{ff}(\kappa_x, \kappa_y) &= \frac{1}{(2\pi)^2} \iint R_{ff}(\xi_x, \xi_y) \left(\begin{array}{l} 1 - \dfrac{\left(\kappa_x \xi_x + \kappa_y \xi_y\right)^2}{2!} + \\ \dfrac{\left(\kappa_x \xi_x + \kappa_y \xi_y\right)^4}{4!} + \cdots \end{array} \right) d\xi_x d\xi_y \\
&= \frac{1}{(2\pi)^2} \iint R_{ff}(\xi_x, \xi_y) \left(\begin{array}{l} 1 - \dfrac{1}{2!} \left(\begin{array}{l} \left(\kappa_x \xi_x\right)^2 + \left(\kappa_y \xi_y\right)^2 + \\ 2\kappa_x \xi_x \kappa_y \xi_y \end{array} \right) + \\ \dfrac{1}{4!} \left(\begin{array}{l} \left(\kappa_x \xi_x\right)^4 + \left(\kappa_y \xi_y\right)^4 + \\ 6\left(\kappa_x \xi_x \kappa_y \xi_y\right)^2 + \\ 4\left(\kappa_x \xi_x\right)^3 \left(\kappa_y \xi_y\right) + \\ 2\left(\kappa_y \xi_y\right)^3 \left(\kappa_x \xi_x\right) + \end{array} \right) + \cdots \end{array} \right) d\xi_x d\xi_y
\end{aligned}
\tag{7.1-4a}
$$

If the process is quadrant symmetry, i.e., $R_{ff}(\xi_x,\xi_y)=R_{ff}(\xi_x,-\xi_y)$, Eq. (7.1-4a) is expressed as,

$$S_{ff}(\kappa_x,\kappa_y)=\frac{1}{(2\pi)^2}\iint R_{ff}(\xi_x,\xi_y)\begin{pmatrix}1-\dfrac{1}{2!}\left[\left(\kappa_x\xi_x\right)^2+\left(\kappa_y\xi_y\right)^2\right]+\\ \dfrac{1}{4!}\begin{bmatrix}\left(\kappa_x\xi_x\right)^4+\left(\kappa_y\xi_y\right)^4+\\ 6\left(\kappa_x\xi_x\kappa_y\xi_y\right)^2\end{bmatrix}+\cdots\end{pmatrix}d\xi_x d\xi_y \quad (7.1\text{-}4b)$$

To simplify the analysis that follows, we will consider the quadrant symmetric process.

From Eqs. (7.1-2) and (7.1-4b), the conditions of homogeneity of the integral processes $F_x(x,y)$, $F_y(x,y)$ and $F(x,y)$ are summarized as follows,

Case 1: $S_{ff}(0,0)\neq 0$ $F_x(x,y)$, $F_y(x,y)$ and $F(x,y)$ are all nonhomogeneous.

Case 2: $S_{ff}(0,0)=S_{ff}^{xx}(0,0)=S_{ff}^{yy}(0,0)=0$ $F(x,y)$ is homogeneous.

Case 3: $S_{ff}(0,0)=S_{ff}^{yy}(0,0)=0,\text{and}\, S_{ff}^{xx}(0,0)\neq 0$ $F_x(x,y)$ is homogeneous.

Case 4: $S_{ff}(0,0)=S_{ff}^{xx}(0,0)=0,\text{and}\, S_{ff}^{yy}(0,0)\neq 0$ $F_y(x,y)$ is homogeneous.

where $S_{ff}^{xx}(0,0)$ and $S_{ff}^{yy}(0,0)$ are the second derivative values of $S_{ff}(\kappa_x,\kappa_y)$ at the origin given by

$$S_{ff}^{xx}(0,0) = \left. \frac{\partial^2 S_{ff}(\kappa_x, \kappa_y)}{\partial \kappa_x^2} \right|_{\kappa_x=\kappa_y=0}$$
$$= -\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi_x^2 R_{ff}(\xi_x, \xi_y) d\xi_x d\xi_y$$
$$S_{ff}^{yy}(0,0) = \left. \frac{\partial^2 S_{ff}(\kappa_x, \kappa_y)}{\partial \kappa_y^2} \right|_{\kappa_x=\kappa_y=0} \tag{7.1-5}$$
$$= -\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi_y^2 R_{ff}(\xi_x, \xi_y) d\xi_x d\xi_y$$

Similar to Eq. (2.14a), the variance σ_A^2 of $f_A(x,y)$ is given such that,

$$\sigma_A^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{\sin \frac{\kappa_x D_x}{2}}{\frac{\kappa_x D_x}{2}} \right)^2 \left(\frac{\sin \frac{\kappa_y D_y}{2}}{\frac{\kappa_y D_y}{2}} \right)^2 S_{ff}(\kappa_x, \kappa_y) d\kappa_x d\kappa_y \tag{7.1-6}$$
$$= \frac{1}{A} \int_{-D_x}^{D_x} \int_{-D_y}^{D_y} \left(1 - \frac{|\xi_x|}{D_x} \right) \left(1 - \frac{|\xi_y|}{D_y} \right) R_{ff}(\xi_x, \xi_y) d\xi_x d\xi_y$$

For Case 2 where $F(x,y)$ is homogeneous, Eq. (7.1-6) becomes

$$\sigma_A^2 = \frac{16}{A^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{S_{ff}(\kappa_x, \kappa_y)}{\kappa_x^2 \kappa_y^2} \left(\sin \frac{\kappa_x D_x}{2} \right)^2 \left(\sin \frac{\kappa_y D_y}{2} \right)^2 d\kappa_x d\kappa_y$$
$$= \frac{16}{A^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{FF}(\kappa_x, \kappa_y) \left(\frac{1 - \cos \kappa_x D_x}{2} \right) \left(\frac{1 - \cos \kappa_y D_y}{2} \right) d\kappa_x d\kappa_y \tag{7.1-7a}$$
$$= \frac{4}{A^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{FF}(\kappa_x, \kappa_y) \begin{pmatrix} 1 - \cos \kappa_x D_x - \cos \kappa_y D_y + \\ \cos \kappa_x D_x \cos \kappa_y D_y \end{pmatrix} d\kappa_x d\kappa_y$$
$$= \frac{4}{A^2} \left(R_{FF}(0,0) + R_{FF}(D_x, D_y) - R_{FF}(D_x, 0) - R_{FF}(0, D_y) \right)$$

In the derivation of Eq. (7.1-7), the following Wiener Kintchine relationships for a quadrant symmetric stochastic fields is used,

$$S_{ff}(\kappa_x,\kappa_y)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}R_{ff}(\xi_x,\xi_y)\cos\kappa_x\xi_x\cos\kappa_y\xi_y d\xi_x d\xi_y$$

$$R_{ff}(\xi_x,\xi_y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}S_{ff}(\kappa_x,\kappa_y)\cos\kappa_x\xi_x\cos\kappa_y\xi_y d\kappa_x d\kappa_y \tag{7.1-7b}$$

For Case 3 where $F_x(x,y)$ is homogeneous, the variance σ_A^2 given by Eq. (7.1-6) is

$$\begin{aligned}\sigma_A^2&=\left(\frac{2}{D_x}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{S_{ff}(\kappa_x,\kappa_y)}{\kappa_x^2}\left(\sin\frac{\kappa_x D_x}{2}\right)^2\left(\frac{\sin\frac{\kappa_y D_y}{2}}{\frac{\kappa_y D_y}{2}}\right)^2 d\kappa_x d\kappa_y\\&=\left(\frac{\sqrt{2}}{D_x}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}S_{F_xF_x}(\kappa_x,\kappa_y)\left(1-\cos\kappa_x D_x\right)\left(\frac{\sin\frac{\kappa_y D_y}{2}}{\frac{\kappa_y D_y}{2}}\right)^2 d\kappa_x d\kappa_y\end{aligned} \tag{7.1-8}$$

For Case 4 where $F_y(x,y)$ is homogeneous, the variance σ_A^2 given by Eq. (7.1-6) is

$$\sigma_A^2=\left(\frac{\sqrt{2}}{D_y}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}S_{F_xF_x}(\kappa_x,\kappa_y)\left(1-\cos\kappa_y D_y\right)\left(\frac{\sin\frac{\kappa_x D_x}{2}}{\frac{\kappa_x D_x}{2}}\right)^2 d\kappa_x d\kappa_y \tag{7.1-9}$$

7.2 Definitions of Correlation Scales of Two Dimensional Processes

When $D_x = D_y \to 0$, the variance σ_A^2 given by Eq. (7.1-6) approaches σ_{ff}^2 since the window function in Eq. (7.1-6) approaches one. That is,

$$\sigma_A^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{ff}(\kappa_x,\kappa_y)d\kappa_x d\kappa_y = \sigma_{ff}^2 \quad \text{for } D_x = D_y \to 0 \tag{7.2-1}$$

On the other hand, for $D_x = D_y \to \infty$, the variance σ_A^2 takes the following forms depending on the behavior of the power spectral density function $S_{ff}(\kappa_x,\kappa_y)$ of $f(x,y)$ at the origin.

For Case 1: from Eq. (7.1-6) when $D_x = D_y \to \infty$,

$$\begin{aligned}\sigma_A^2 &= \frac{4}{D_x D_y}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin u}{u}\right)^2\left(\frac{\sin v}{v}\right)^2 S_{ff}\left(\frac{2u}{D_x},\frac{2v}{D_y}\right)dudv \\ &= \frac{4}{D_x D_y}S_{ff}(0,0)\int_{-\infty}^{\infty}\left(\frac{\sin u}{u}\right)^2 du\int_{-\infty}^{\infty}\left(\frac{\sin v}{v}\right)^2 dv \\ &= \frac{(2\pi)^2}{D_x D_y}S_{ff}(0,0) \\ &= \frac{1}{D_x D_y}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{ff}(\xi_x,\xi_y)d\xi_x d\xi_y\end{aligned} \tag{7.2-2a}$$

For Case 2: from Eq. (7.1-7a),

$$\sigma_A^2 = \frac{4}{(D_x D_y)^2}R_{FF}(0,0) = \frac{4}{(D_x D_y)^2}\sigma_{FF}^2 \quad \text{for } D_x = D_y \to \infty \tag{7.2-2b}$$

where σ_{FF}^2 is the variance of $F(x,y)$.

For Case 3: from Eq. (7.1-8),

$$\sigma_A^2 = \left(\frac{\sqrt{2}}{D_x}\right)^2 \frac{2}{D_y} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{F_xF_x}\left(\kappa_x, \frac{2v}{D_y}\right)\left(1-\cos\kappa_x D_x\right)\left(\frac{\sin v}{v}\right)^2 d\kappa_x dv$$

$$= \left(\frac{\sqrt{2}}{D_x}\right)^2 \frac{2}{D_y} \int_{-\infty}^{\infty} S_{F_xF_x}\left(\kappa_x, 0\right)\left(1-\cos\kappa_x D_x\right) d\kappa_x \int_{-\infty}^{\infty}\left(\frac{\sin v}{v}\right)^2 dv$$

(7.2-2c)

$$= \left(\frac{\sqrt{2}}{D_x}\right)^2 \frac{2\pi}{D_y} \int_{-\infty}^{\infty} S_{F_xF_x}\left(\kappa_x, 0\right)\left(1-\cos\kappa_x D_x\right) d\kappa_x$$

$$= \left(\frac{\sqrt{2}}{D_x}\right)^2 \frac{2\pi}{D_y} R_{F_xF_x}(0) \quad \text{for } D_x = D_y \to \infty$$

where $R_{F_xF_x}(\xi_x)$ is defined such that,

$$R_{F_xF_x}(\xi_x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{F_xF_x}(\xi_x, \xi_y) d\xi_y \qquad (7.2\text{-}2d)$$

then

$$S_{F_xF_x}(\kappa_x, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{F_xF_x}(\xi_x)\cos\kappa_x\xi_x d\xi_x \qquad (7.2\text{-}2e)$$

The inverse transform reclaims

$$R_{F_xF_x}(\xi_x) = \int_{-\infty}^{\infty} S_{F_xF_x}(\kappa_x, 0)\cos\kappa_x\xi_x d\kappa_x \qquad (7.2\text{-}2f)$$

For Case 4: from Eq. (7.1-9),

$$\sigma_A^2 = \left(\frac{\sqrt{2}}{D_y}\right)^2 \frac{2\pi}{D_x} R_{F_yF_y}(0) \quad \text{for } D_x = D_y \to \infty \qquad (7.2\text{-}2g)$$

where $R_{F_xF_x}(\xi_y)$ is defined by,

$$R_{F_xF_x}(\xi_y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{F_yF_y}(\xi_x,\xi_y)d\xi_x$$
$$= \int_{-\infty}^{\infty} S_{F_yF_y}(0,\kappa_y)\cos\kappa_y\xi_y d\kappa_y \qquad (7.2\text{-}2h)$$

and

$$S_{F_yF_y}(0,\kappa_y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{F_yF_y}(\xi_y)\cos\kappa_y\xi_y d\xi_y \qquad (7.2\text{-}2i)$$

Summarizing the above Eqs. (7.2-1) and (7.2-2), the correlation scales of $f(x,y)$ can be defined as follows.

For Case 1:

$$\frac{\sigma_A^2}{\sigma_{ff}^2} = \begin{cases} 1 & D_x = D_y \to 0 \\ \dfrac{A^*}{A(=D_xD_y)} & D_x = D_y \to \infty \end{cases} \qquad (7.2\text{-}3a)$$

where A^* is the correlation scale of for Case 1 defined by

$$A^* = \frac{1}{\sigma_{ff}^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{ff}\left(\xi_x,\xi_y\right)d\xi_x d\xi_y = \frac{(2\pi)^2}{\sigma_{ff}^2} S_{ff}\left(0,0\right) \qquad (7.2\text{-}3b)$$

For Case 2:

$$\frac{\sigma_A^2}{\sigma_{ff}^2} = \begin{cases} 1 & D_x = D_y \to 0 \\ \left(\dfrac{A_F^*}{A}\right)^2 & D_x = D_y \to \infty \end{cases} \qquad (7.2\text{-}4a)$$

where A_F^* is the correlation scale of for Case 2 defined by

$$A_F^* = 2\frac{\sigma_{FF}}{\sigma_{ff}} = 2\frac{1}{(2\pi)^2}A_F$$

$$A_F = (2\pi)^2 \frac{\sigma_{FF}}{\sigma_{ff}}$$

$$= (2\pi)^2 \sqrt{\frac{R_{FF}(0,0)}{\ddot{\ddot{R}}_{FF}(0,0)}}$$

$$= (2\pi)^2 \sqrt{\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{FF}(\kappa_x,\kappa_y)d\kappa_x d\kappa_y}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \kappa_x^2\kappa_y^2 S_{FF}(\kappa_x,\kappa_y)d\kappa_x d\kappa_y}} \tag{7.2-4b}$$

$$\ddot{\ddot{R}}_{FF}(0,0) = \left.\frac{\partial^4 R_{FF}(\xi_x,\xi_y)}{\partial\xi_x^2\partial\xi_y^2}\right|_{\xi_x=\xi_y=0}$$

For Case 3:

$$\frac{\sigma_A^2}{\sigma_{ff}^2} = \begin{cases} 1 & D_x = D_y \to 0 \\ \dfrac{L_{xy}^{*3}}{D_x^2 D_y\left(= D_{xy}^3\right)} & D_x = D_y \to \infty \end{cases} \tag{7.2-5a}$$

where L_{xy}^* is the correlation scale of for Case 3 defined by

$$L_{xy}^* = \left(\frac{2}{\sigma_{ff}^2}\int_{-\infty}^{\infty} R_{F_xF_x}(0,\xi_y)d\xi_y\right)^{1/3}$$

$$= \left(4\pi\frac{\int_{-\infty}^{\infty} R_{F_xF_x}(\kappa_x,0)d\kappa_x}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \kappa_x^2 S_{F_xF_x}(\kappa_x,\kappa_y)d\kappa_x d\kappa_y}\right)^{1/3} \tag{7.2-5b}$$

For Case 4:

$$\frac{\sigma_A^2}{\sigma_{ff}^2} = \begin{cases} 1 & D \to 0 \\ \dfrac{L_{yx}^{*3}}{D_x D_y^2\left(= D_{yx}^3\right)} & D \to \infty \end{cases} \tag{7.2-6a}$$

where L^*_{yx} is the correlation scale of for Case 3 defined by

$$L^*_{yx} = \left(\frac{2}{\sigma^2_{ff}} \int_{-\infty}^{\infty} R_{F_yF_y}(\xi_x, 0) d\xi_x \right)^{1/3}$$

$$= \left(4\pi \frac{\int_{-\infty}^{\infty} R_{F_yF_y}(0, \kappa_y) d\kappa_y}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \kappa_y^2 S_{F_xF_x}(\kappa_x, \kappa_y) d\kappa_x d\kappa_y} \right)^{1/3} \qquad (7.2\text{-}6b)$$

If we consider the special case where $D_x = D_y = D$, the above results become so simple that the following results may be useful for the estimation of the correlation scales of $f(x, y)$ using the graphical method indicated in chapter 6 for one dimensional stochastic processes.

For Case 1:

$$\frac{\sigma_A^2}{\sigma_{ff}^2} = \begin{cases} 1 & D \to 0 \\ \left(\dfrac{L^*}{D} \right)^2 & D \to \infty \end{cases} \qquad (7.2\text{-}7a)$$

where L^* is the equivalent correlation distance of the correlation scale A^* of $f(x, y)$ defined by

$$L^* = \sqrt{A^*} \qquad (7.2\text{-}7b)$$

For Case 2:

$$\frac{\sigma_A^2}{\sigma_{ff}^2} = \begin{cases} 1 & D \to 0 \\ \left(\dfrac{L_F^*}{D} \right)^4 & D \to \infty \end{cases} \qquad (7.2\text{-}8a)$$

where L^*_F is also the equivalent correlation distance of the correlation scale A^*_F

of $f(x,y)$ defined by

$$L_F^* = \sqrt{A_F^*} \tag{7.2-8b}$$

For Case 3:

$$\frac{\sigma_A^2}{\sigma_{ff}^2} = \begin{cases} 1 & D \to 0 \\ \left(\dfrac{L_{xy}^*}{D}\right)^3 & D \to \infty \end{cases} \tag{7.2-9}$$

where L_{xy}^* is the correlation scale of for Case 3 defined by Eq. (7.2-5b).

For Case 4:

$$\frac{\sigma_A^2}{\sigma_{ff}^2} = \begin{cases} 1 & D \to 0 \\ \left(\dfrac{L_{yx}^*}{D}\right)^3 & D \to \infty \end{cases} \tag{7.2-10}$$

where L_{yx}^* is the correlation scale of for Case 4 defined by Eq. (7.2-6b).

Although Eqs. (7.2-9) and (7.2-10) have the same form with respect to the separation distance D, it is easy to distinguish them. In fact, if we select a rectangular area where $D_x = 4D_y = D$ for example, from Eqs. (7.2-5a) and (7.2-6a), a difference appears between Eqs. (7.2-9) and (7.2-10) such that $4(L_{xy}^* / D)^3$ for Case 3 and $16(L_{yx}^* / D)^3$ for Case 4. By this difference, we can distinguish between Case 3 and Case 4.

In Fig. 7.2-1, the approximate relationships between σ_A and D given by Eqs. (7.2-7) to (7.2-10) are shown by solid lines. In the same figure, three examples of exact relationships Eqs. (7.2-11b), (7.2-12b) and (7.2-13b) are plotted (dashed curves) using the following particular forms (separable types) of the correlation function or power spectral density function which satisfy quadrant symmetric condition.

It can be observed from Fig. 7.2-1 that all these curves asymptotically approach the solid lines in the ranges where $D \to 0$ and $D \to \infty$, and also that in the intermediate range of D, the solid lines tend to represent the upper bound of all the dashed curves.

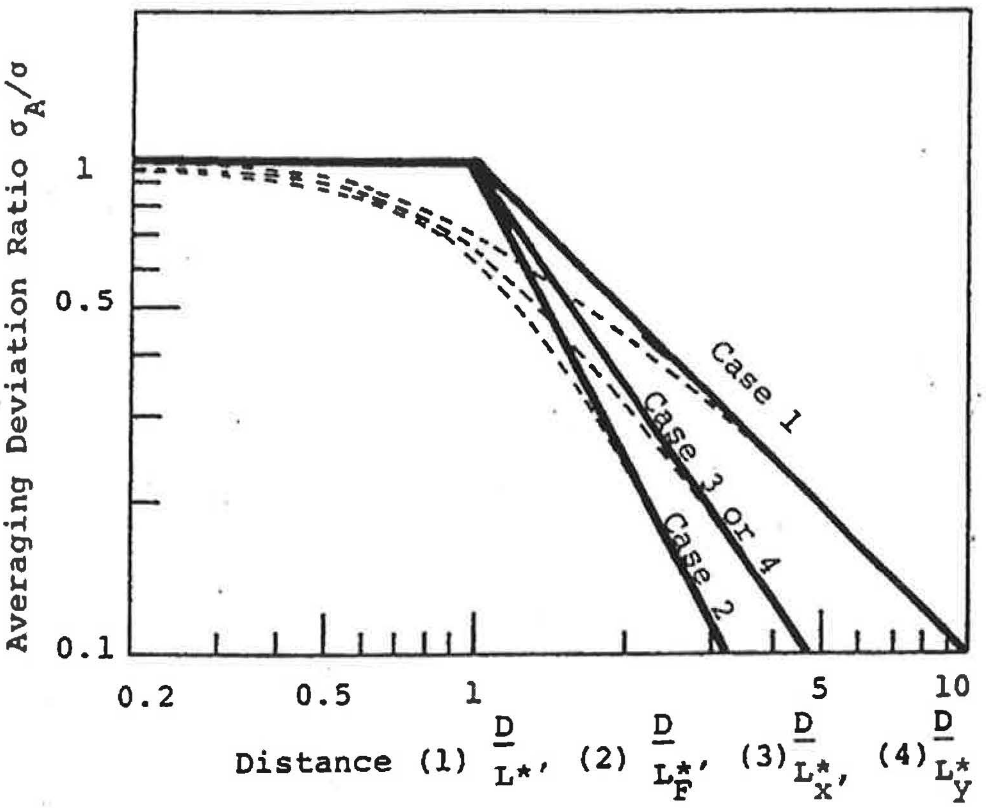


Fig. 7.2-1 $\sigma_A - D$ Diagram for Two Dimensional Process Where $D = D_x = D_y$
(Solid Lines: Approximation, Dashed Curves: Exact Solutions)

In the following examples, we assume $D_x = D_y = D, b_x = b_y = b$.

Power spectrum for Case 1:

$$\begin{aligned} S_{ff}(\kappa_x, \kappa_y) &= \sigma_{ff}^2 S(\kappa_x) S(\kappa_y) \\ &= \frac{\sigma_{ff}^2}{4\pi} b^2 \exp\left(-\left(\frac{b\kappa_x}{2}\right)^2 - \left(\frac{b\kappa_y}{2}\right)^2\right) \end{aligned} \tag{7.2-11a}$$

The exact variation is obtained from Eq. (7.1-6) such that

$$\begin{aligned} \left(\frac{\sigma_A}{\sigma_{ff}}\right)^2 &= \left(\frac{2}{D}\int_0^D \left(1 - \frac{\xi}{D}\right) R_{ff}(\xi) d\xi\right)^2 \\ &= \left(\sqrt{\pi}\left(\frac{b}{D}\right) B\left(\frac{\sqrt{2}D}{b}\right) + \left(\frac{b}{D}\right)^2 \left(\exp\left(-\left(\frac{D}{b}\right)^2\right) - 1\right)\right)^2 \end{aligned} \tag{7.2-11b}$$

$$R_{ff}(\xi) = \mathrm{e}^{-(\xi/b)^2}, \; b = \frac{L^*}{\sqrt{\pi}}$$

$$B\left(\frac{\sqrt{2}D}{b}\right) = 2\Phi\left(\frac{\sqrt{2}D}{b}\right) - 1$$

where the standard normal distribution $\Phi(x)$ is used as,

$$\begin{aligned} \Phi(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^x \exp\left(-\frac{t^2}{2}\right) dt \\ &= \frac{1}{2} + \frac{x}{\sqrt{2\pi}}\left(1 - \frac{x^2}{3 \cdot 2 \cdot 1!} + \frac{x^4}{5 \cdot 2^2 \cdot 2!} - \frac{x^6}{7 \cdot 2^3 \cdot 3!} + \cdots\right) \end{aligned} \tag{7.2-11c}$$

and

$$\begin{aligned} \int_0^D \mathrm{e}^{-(\xi/b)^2} d\xi &= \frac{b}{\sqrt{2}} \int_0^{\sqrt{2}D/b} \mathrm{e}^{-t^2/2} dt \\ &= \frac{b\sqrt{\pi}}{2}\left(2\Phi\left(\frac{\sqrt{2}D}{b}\right) - 1\right) \\ &= \frac{b\sqrt{\pi}}{2} B\left(\frac{\sqrt{2}D}{b}\right) \end{aligned} \tag{7.2-11d}$$

Power spectrum for Case 2:

$$S_{ff}(\kappa_x,\kappa_y)=\frac{\sigma_{ff}^2}{16\pi}b^6\kappa_x^2\kappa_y^2\exp\left(-\left(\frac{b\kappa_x}{2}\right)^2-\left(\frac{b\kappa_y}{2}\right)^2\right)$$

$$S_{FF}(\kappa_x,\kappa_y)=\frac{S_{ff}(\kappa_x,\kappa_y)}{\kappa_x^2\kappa_y^2}=\frac{\sigma_{ff}^2}{16\pi}b^6\exp\left(-\left(\frac{b\kappa_x}{2}\right)^2-\left(\frac{b\kappa_y}{2}\right)^2\right)$$

$$R_{FF}(\xi_x,\xi_y)=\frac{\sigma_{ff}^2}{4}b^4\exp\left(-\left(\frac{b\xi_x}{2}\right)^2-\left(\frac{b\xi_y}{2}\right)^2\right) \quad (7.2\text{-}12a)$$

$$A_F^*=b^2,\ L_F^*=\sqrt{A_F^*}=b$$

The exact variation is obtained from Eq. (7.1-7a) such as

$$\begin{aligned}\left(\frac{\sigma_A}{\sigma_{ff}}\right)^2&=\frac{4}{A^2}\begin{pmatrix}R_{FF}(0,0)+R_{FF}(D_x,D_y)-\\ R_{FF}(D_x,0)-R_{FF}(0,D_y)\end{pmatrix}\\ &=\left(\left(\frac{L_F^*}{D}\right)^2\left(1-\exp\left(-\left(\frac{D}{L_F^*}\right)^2\right)\right)\right)^2\end{aligned} \quad (7.2\text{-}12b)$$

Power spectrum for Case 3:

$$\begin{aligned}S_{ff}(\kappa_x,\kappa_y)&=\sigma_{ff}^2S_{F_xF_x}(\kappa_x)S_{ff}(\kappa_y)\\ &=\frac{\sigma_{ff}^2}{8\pi}b^4\kappa_x^2\exp\left(-\left(\frac{b\kappa_x}{2}\right)^2-\left(\frac{b\kappa_y}{2}\right)^2\right)\end{aligned}$$

$$S_{F_xF_x}(\kappa_x)=\frac{b^3}{4\sqrt{\pi}}\exp\left(-\left(\frac{b\kappa_x}{2}\right)^2\right),\ S_{ff}(\kappa_y)=\frac{b}{2\sqrt{\pi}}\exp\left(-\left(\frac{b\kappa_y}{2}\right)^2\right) \quad (7.2\text{-}13a)$$

$$R_{F_xF_x}(\xi_x)=\frac{b^2}{2}\exp\left(-\left(\frac{b\xi_x}{2}\right)^2\right),\ R_{ff}(\xi_y)=\exp\left(-\left(\frac{\xi_y}{b}\right)^2\right)$$

$$b=\frac{L_{xy}^*}{\left(\sqrt{\pi}\right)^{1/3}}$$

The exact variation is obtained from Eq. (7.1-8) such as

$$\left(\frac{\sigma_A}{\sigma_{ff}}\right)^2 = \left(\frac{2}{D^2}\left(R_{F_xF_x}(0) - R_{F_xF_x}(D)\right)\right)\left(\frac{2}{D}\int_0^D\left(1-\frac{\xi}{D}\right)R_{ff}(\xi)d\xi\right)$$

$$= \left(\left(\frac{b}{D}\right)^2\left(1-\exp\left(-\left(\frac{D}{b}\right)^2\right)\right)\right)\times$$

$$\left(\sqrt{\pi}\left(\frac{b}{D}\right)B\left(\frac{\sqrt{2}D}{b}\right)+\left(\frac{b}{D}\right)^2\left(\exp\left(-\left(\frac{D}{b}\right)^2\right)-1\right)\right) \qquad (7.2\text{-}13b)$$

$$B\left(\frac{\sqrt{2}D}{b}\right) = 2\Phi\left(\frac{\sqrt{2}D}{b}\right)-1$$

<u>Power spectrum for Case 4:</u>

This Case is same form of case 3 by changing the coordinates x, y. However, since $D_x = D_y = D, b_x = b_y = b$ is assumed in this example, The exact variation is same to Case 3.

$$S_{ff}(\kappa_x,\kappa_y) = \frac{\sigma_{ff}^2}{8\pi}b^4\kappa_y^2\exp\left(-\left(\frac{b\kappa_x}{2}\right)^2-\left(\frac{b\kappa_y}{2}\right)^2\right)$$

$$S_{F_yF_y}(\kappa_y) = \frac{b^3}{4\sqrt{\pi}}\exp\left(-\left(\frac{b\kappa_y}{2}\right)^2\right),\ S_{ff}(\kappa_x) = \frac{b}{2\sqrt{\pi}}\exp\left(-\left(\frac{b\kappa_x}{2}\right)^2\right)$$

$$R_{F_yF_y}(\xi_y) = \frac{b^2}{2}\exp\left(-\left(\frac{b\xi_y}{2}\right)^2\right),\ R_{ff}(\xi_x) = \exp\left(-\left(\frac{\xi_x}{b}\right)^2\right) \qquad (7.2\text{-}14)$$

$$b = \frac{L_{yx}^* = L_{xy}^*}{\left(\sqrt{\pi}\right)^{1/3}}$$

7.3 Numerical Examples

In order to visually illustrate the correlation scales and patterns of spatial variation of $f(x,y)$, we present in this section some numerical examples simulated by the following equations for quadrant symmetric processes (Shinozuka and Harada (1986)).

$$f(x,y)=\sqrt{2}\sum_{m=1}^{M}\sum_{n=1}^{N}\sqrt{2S_{ff}\left(\kappa_{x_m},\kappa_{y_n}\right)d\kappa_x d\kappa_y}\times\left\{\begin{matrix}\cos\left(\kappa_{x_m}x+\kappa_{y_n}y+\theta_{1mn}\right)+\\ \cos\left(\kappa_{x_m}x-\kappa_{y_n}y+\theta_{2mn}\right)\end{matrix}\right\} \tag{7.3-1}$$

$$d\kappa_x=\frac{\kappa_{xu}}{M},\quad d\kappa_y=\frac{\kappa_{yu}}{N},$$
$$\kappa_{x_m}=md\kappa_x,\quad \kappa_{y_n}=nd\kappa_y$$

where θ_{1mn} and θ_{2mn} are independent random phase angles uniformly distributed between 0 and 2π. The parameters κ_{xu} and κ_{yu} are the upper cut off wave numbers of κ_x and κ_y, respectively.

Example 1: This example is for case 1 using the power spectrum given by Eq. (7.2-11a) together with following data;

$$\sigma_{ff}=1\,\mathrm{m},\quad b_x=1\,\mathrm{m},\quad b_y=1/\sqrt{2}\,\mathrm{m}$$
$$M=N=64,\quad \kappa_{xu}=\kappa_{yu}=2\pi\,\mathrm{rad}/\mathrm{m}$$

A sample function of $f(x,y)$ and the size of the correlation area A^* in this example are shown in Fig. 7.3-1. In Case 1, the correlation area A^* defined by Eq. (7.2-3b) may signify that the correlation of is extremely high within the size of this area $A^*(=\pi b_x b_y=2.22\ \mathrm{m}^2)$.

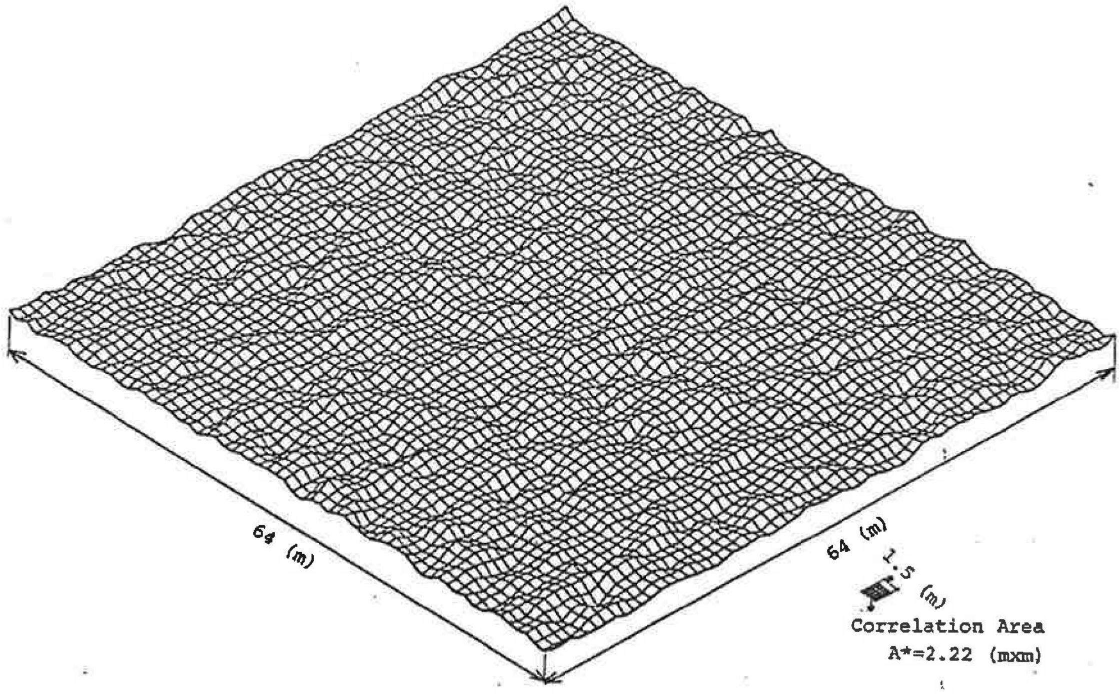


Fig. 7.3-1 Sample Function of $f(x,y)$ for Case 1

Example 2: For Case 2 using the power spectrum of Eq. (7.2-12a) in conjunction with the following data;

$$\sigma_{ff} = 1\,\mathrm{m}, \quad b_x = b_y = \pi / 5\,\mathrm{m}$$
$$M = N = 64, \quad \kappa_{xu} = \kappa_{yu} = 4\pi\,\mathrm{rad/m}$$

In Fig. 7.3-2, the size of the correlation area $A_F^*(= b_x b_y = 0.4\ \mathrm{m}^2)$ in this example and a sample function are shown. For Case 2, the correlation area A_F^* defined by Eq. (7.2-4b) may also be useful for representing the size of area within which highly correlated observations are made.

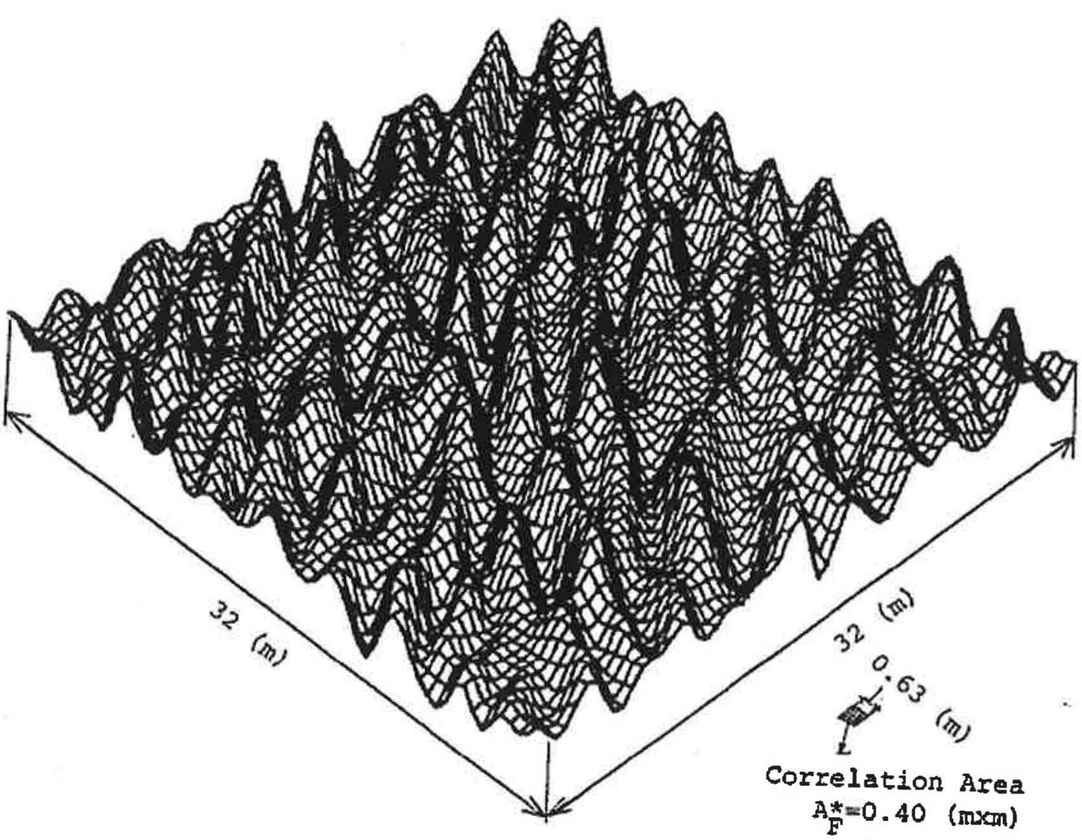


Fig. 7.3-2 Sample Function of $f(x,y)$ for Case 2

Example 3: This is an example for case 3 using the power spectrum given by Eq. (7.2-13a) with following data;

$$\sigma_{ff} = 1.24\,\mathrm{cm}, \quad b_x = 1.131\times10^3\,\mathrm{m}, \quad b_y = 3.012\times10^3\,\mathrm{m}$$
$$M = N = 64, \quad \kappa_{xu} = 10\,/\,b_x = 8.84\times10^{-3}\,\mathrm{rad\,/\,m}$$
$$\kappa_{yu} = 10\,/\,b_y = 3.32\times10^{-3}\,\mathrm{rad\,/\,m}$$

The size of the correlation distance $L^*_{xy}(=\left(\sqrt{\pi}b_x^2b_y\right)^{1/3} = 1897.4\,\mathrm{m})$ in this example and a sample function of $f(x,y)$ are shown in Fig. 7.3-3. In this case, relatively rapid variation along the x axis is observed, compared with the variation along the y axis. To represent this variation along the x axis, the correlation distance L_{xy}^* defined by Eq. (7.2-5b) may be appropriate. However, the correlation distance along the y axis is not defined, and the size of correlation area $A^*(= L_{xy}^{*2})$ is quite vague in Case 3.

For the separable power spectrum of this example, the correlation distances along the x, y axes can be defined such as,

$$L_{F_x}^* = b_x = 1.131 \times 10^3 \text{ m}, \quad L_y^* = \sqrt{\pi} b_y = 5.448 \times 10^3 \text{ m} \tag{7.3-2a}$$

Using the above definition, the correlation distance L_{xy}^* is expressed as,

$$\begin{aligned} L_{xy}^* &= \left(\sqrt{\pi} b_x^2 b_y\right)^{1/3} = \left(L_{F_x}^{*2} L_y^*\right)^{1/3} \\ &= \left((1132)^2 \times 5448\right)^{1/3} = 1897.4 \text{ m} \end{aligned} \tag{7.3-2b}$$

The correlation distance L_{xy}^* is about 1.7 times longer than the correlation distance $L_{F_x}^*$. By observing the variation in Fig. 7.3-3, it may be suitable to use the correlation distance $L_{F_x}^*$ along the x axis and the correlation distance L_y^* along the y axis, respectively.

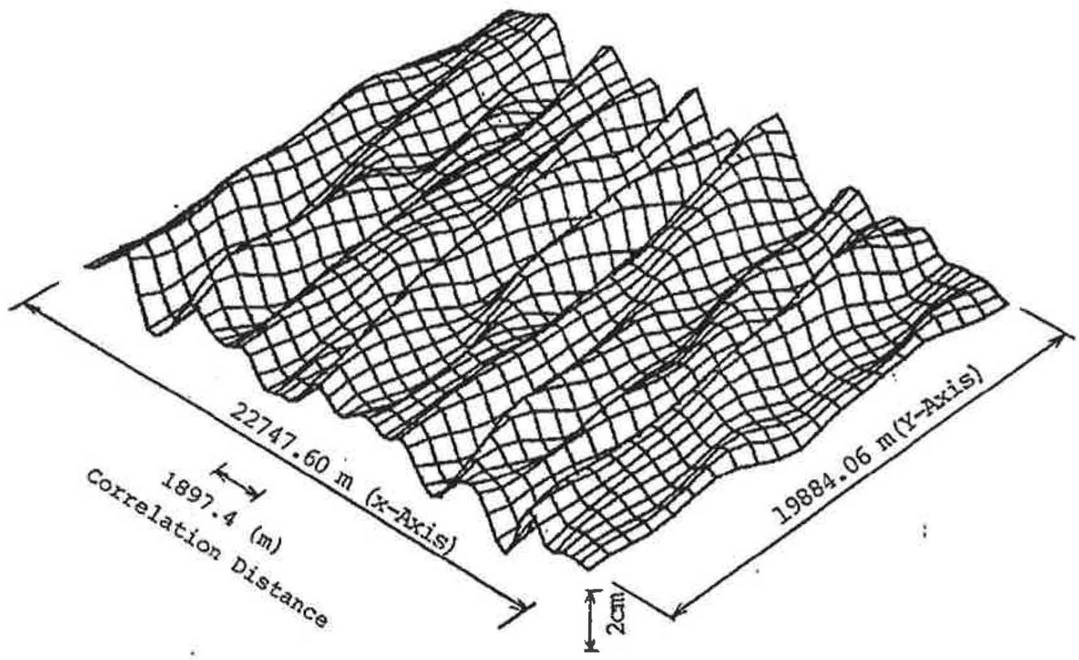


Fig. 7.3-3 Sample Function of $f(x,y)$ for Case 3

8. SOME NEW APPLICATION EXAMPLES OF CORRELATION SCALES

In this chapter, we will show three examples of application of correlation scales; (1) Peak mean factor, (2) Seismic relative ground displacement between two locations (Seismic ground root mean square values between two points), (3) Upper cut off wavenumber of power spectrum density function.

8.1 Peak Mean Factor

As briefly described in Section 1.2, the correlation scale has been successfully used in many engineering fields as a measure of approximately obtaining the equivalent number of independent observations from stochastic process data with finite intervals. In the context of this interpretation of correlation scale, we present here a new approximate observation of the probability distribution of maximum values of stochastic processes.

In many applications of stochastic process theory to the analysis and design of structures, a central question is as follows: What is the absolute maximum value of $f(x)$ with zero mean over the range $0 \leq x \leq L$ where the correlation function or the power spectral density function is known. If the absolute maximum value Y is expressed as $p\sigma_{ff}$, where σ_{ff} is the standard deviation of $f(x)$, and p is the peak stochastic factor, the mean and standard deviation of p is given by Davenport (1964) as,

$$E[p]=\sqrt{2\ln\left(\frac{2L}{L_f}\right)}+\frac{0.577215}{\sqrt{2\ln\left(\frac{2L}{L_f}\right)}}$$

$$\sigma_{pp}=\frac{\pi}{\sqrt{6}}\frac{1}{\sqrt{2\ln\left(\frac{2L}{L_f}\right)}} \tag{8.1-1a}$$

where L_f is the apparent wave length defined by

$$L_f=2\pi\frac{\sigma_{ff}}{\sigma_{\dot{f}\dot{f}}}=2\pi\sqrt{-\frac{R_{ff}(0)}{R''_{ff}(0)}}=2\pi\sqrt{\frac{\int_{-\infty}^{\infty}S_{ff}(\kappa)d\kappa}{\int_{-\infty}^{\infty}\kappa^2 S_{ff}(\kappa)d\kappa}} \tag{8.1-1b}$$

Equations (8.1-1a) and (8.1-1b) assume the existence of L_f defined by Eq. (8.1-1b). However, the derivative $f'(x)$ of $f(x)$ over does not exist when $S_{ff}(0)\neq 0$ (Case I process). In this case, the above equations are useless. Hence, we need another stable expression for the peak factor. The following equations for peak factors are based on the combination of the largest value distribution function (the first type) and the correlation scales A an C in Table 1-1.

For the probability density function of the local maxima X (local points of a homogeneous Gaussian process, the general expression is well known as (Cartwright and Languet-Higgins (1956)).

$$F_X(y)=\frac{\varepsilon}{\sqrt{2\pi}\sigma}\exp\left(-\frac{y^2}{2\varepsilon^2\sigma^2}\right)+\frac{y\sqrt{1-\varepsilon^2}}{\sigma^2}\exp\left(-\frac{y^2}{2\sigma^2}\right)F_X^N\left(\frac{y\sqrt{1-\varepsilon^2}}{\sigma\varepsilon}\right) \tag{8.1-2}$$

Where $F_X^N(\bullet)$ is a normal distribution function and ε is the irregularity factor and lies between 0 and 1. For $\varepsilon = 0$ (completely narrow band process), the first term vanishes and Eq. (8.1-2) reduces to the Rayleigh distribution such that

$$f_X^R(y) = \frac{y}{\sigma^2}\exp\left(-\frac{y^2}{2\sigma^2}\right) \tag{8.1-3a}$$

with distribution function

$$F_X^R(y) = 1 - \exp\left(-\frac{y^2}{2\sigma^2}\right) \tag{8.1-3b}$$

For $\varepsilon = 1$ (completely wide band process), only the first term remains and Eq. (8.1-2) becomes a Gaussian distribution with zero mean and variance σ^2 such that

$$f_X^N(y) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{y^2}{2\sigma^2}\right) \tag{8.1-4a}$$

with distribution function

$$F_X^N(y) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{y}\exp\left(-\frac{y^2}{2\sigma^2}\right)dy \tag{8.1-4b}$$

The relationship between the local peak probability density function $f_X(y)$ and the homogeneous (ergodic) process $f(x)$ is schematically illustrated in Fig. 8.1-1.

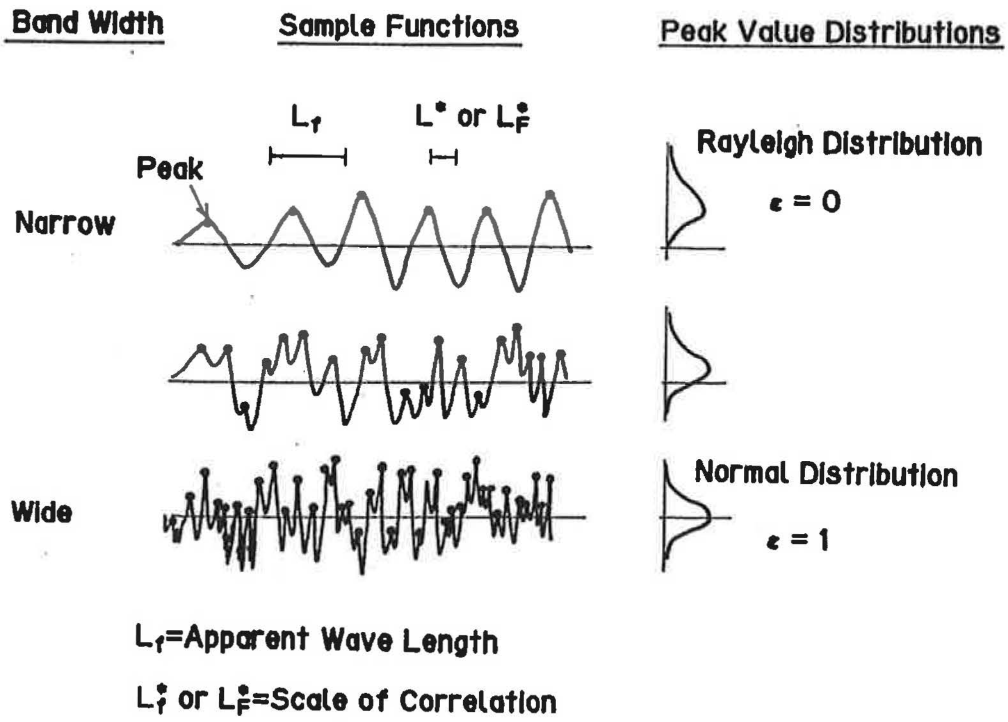


Fig. 8.1-1 Schematic Illustration of Sample Functions and Peal Value Distributions

On the other hand, using the exact distribution function $F_Y(y)$ for the greatest peak values among $(X_1, X_2, \cdots X_n)$ that are statistically independent and identically distributed with $F_X(y)$ as the initial variate such that

$$\begin{aligned} F_Y(y) &= \Pr\left(Y \le y\right) \\ &= \Pr\left(X_1 \le y, X_2 \le y, \cdots, X_n \le y\right) \\ &= \left(F_X(y)\right)^n \end{aligned} \tag{8.1-5}$$

We may have an approximate distribution function for the "greatest peaks" of the stochastic process $f(x)$ over the range $0 \le x \le L$ when in Eq. (8.1-5) we interpret $F_X(y)$ as the distribution function of the distribution density function given by Eq. (8.1-2) with n given as follows:

$$n = \frac{L}{L^*} \quad \text{or} \quad \frac{L}{L_F^*} \tag{8.1-6a}$$

where L^* and L_F^* are the correlation scales (A and C in Table 1-1 or Eq. (3.8)) such that

$$L^* = \frac{1}{\sigma_{ff}^2} \int_{-\infty}^{\infty} R_{ff}(\xi) d\xi \tag{8.1-6b}$$

and

$$\begin{aligned} L_F^* &= \frac{1}{\sqrt{2\pi}} L_F = \frac{1}{\sqrt{2\pi}} \left(2\pi \frac{\sigma_{FF}}{\sigma_{ff}} \right) \\ &= \frac{1}{\sqrt{2\pi}} \left(2\pi \sqrt{-\frac{R_{FF}(0)}{R_{FF}''(0)}} \right) = \frac{1}{\sqrt{2\pi}} \left(2\pi \sqrt{\frac{\int_{-\infty}^{\infty} S_{FF}(\kappa) d\kappa}{\int_{-\infty}^{\infty} S_{ff}(\kappa) d\kappa}} \right) \end{aligned} \tag{8.1-6c}$$

In Eq. (8.1-6a), the number n signifies the equivalent number of independent observations contained in the interval L since the correlation scales L^* and L_F^* are the measures of the highly correlated length of $f(x)$.

Finally, taking into account the fact that the peaks and troughs generally tend to appear as the same number over a finite length, the approximate distribution function $F_Y^A(y)$ for the absolute maximum value of $f(x)$ over the range $0 \le x \le L$ may be given by Eq. (8.1-5) with the following n instead of the n given by Eq. (8.1-6a):

$$n = \frac{2L}{L^*} \quad \text{or} \quad \frac{2L}{L_F^*} \tag{8.1-7}$$

As is well known, for large n and exponential type initial function $F_X(y)$, Eq. (8.1-5) has the Type I asymptotic form classified by Gumbel (1958) such that

$$F_Y(y) = \exp\left(-e^{-\alpha_n(y-u_n)}\right) \tag{8.1-8a}$$

where u_n = the characteristic largest value of the initial variate X and α_n = an inverse measure of the dispersion of Y which are determined by

$$F_X(u_n) = 1 - \frac{1}{n}, \quad \alpha_n = nf_X(u_n) \tag{8.1-8b}$$

The mean value $E(Y)$ and standard deviation σ_{YY} of Y are also given such that

$$\begin{aligned} E(Y) &= u_n + \frac{0.577215}{\alpha_n} \\ \sigma_{YY} &= \frac{\pi}{\alpha_n\sqrt{6}} \end{aligned} \tag{8.1-8c}$$

As the two extreme cases where $F_X(y) = F_X^R(y)$ and $F_X(y) = F_X^N(y)$, Eq. (8.1-8c) becomes as follows.

For a Rayleigh distribution $F_X(y) = F_X^R(y)$:

$$\begin{aligned} E[p] &= \frac{E[Y]}{\sigma_{ff}} = \sqrt{2\ln n} + \frac{0.577215}{\sqrt{2\ln n}} \\ \sigma_{pp} &= \frac{\pi}{\sqrt{6}}\frac{1}{\sqrt{2\ln n}} \end{aligned} \tag{8.1-9a}$$

When $n = 1$, $F_Y(y) = F_X^R(y)$ from Eq. (8.1-5), then from the mean value and standard deviation of a Rayleigh distribution, the mean peak factor and its standard deviation are given by

$$
\begin{aligned}
E[p] &= \frac{E[Y]}{\sigma_{ff}} = \sqrt{\frac{\pi}{2}} \simeq 1.25 \\
\sigma_{pp} &= \frac{\sigma_{YY}}{\sigma_{ff}} = \sqrt{\left(2 - \frac{\pi}{2}\right)} \simeq 0.65
\end{aligned} \tag{8.1-9b}
$$

For a normal distribution $F_X(y) = F_X^N(y)$:

$$
\begin{aligned}
E[p] &= \frac{E[Y]}{\sigma_{ff}} = \sqrt{2\ln n} - \frac{\ln\ln n + \ln 4\pi}{\sqrt{2\ln n}} + \frac{0.577215}{\sqrt{2\ln n}} \\
\sigma_{pp} &= \frac{\sigma_{YY}}{\sigma_{ff}} = \frac{\pi}{\sqrt{6}} \frac{1}{\sqrt{2\ln n}}
\end{aligned} \tag{8.1-10a}
$$

When $n = 1$, $F_Y(y) = F_X^N(y)$ from Eq. (8.1-5), then from the mean value and standard deviation of a normal distribution, the mean peak factor and its standard deviation are given as

$$
\begin{aligned}
E[p] &= \frac{E[Y]}{\sigma_{ff}} = 0 \\
\sigma_{pp} &= \frac{\sigma_{YY}}{\sigma_{ff}} = 1
\end{aligned} \tag{8.1-10b}
$$

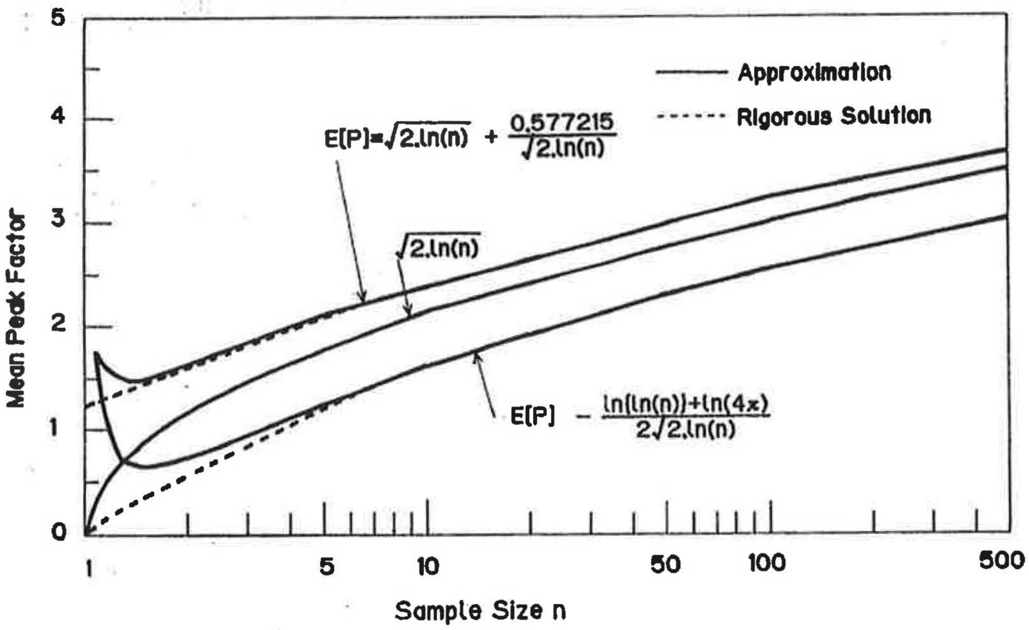


Fig. 8.1-2 Relationships between Mean Peak Factor and Independent Sample Size

The mean peak factors given by Eqs. (8.1-9a) and (8.1-10a) are plotted by solid curves as a function of n in Fig. 8.1-2. The dashed curves in Fig. 8.1-2 are the results from Eq. (8.1-5). From Fig. 8.1-2, Eqs. (8.1-9a) and (8.1-10a) may be used for $n > 10$. For small value of n, Eqs. (8.1-9a) and (8.1-10a) tend to give a larger value of $E[p]$. Since, for the intermediate values of the irregularity factor, the value of $E[p]$ may lie between the two extreme cases (two solid curves and dashed curves in Fig. 8.1-2) with the Rayleigh distribution and normal distributions as the initial distribution, a simpler approximation may be appropriate for the mean value of peak factor:

$$E[p] = \frac{E[Y]}{\sigma_{ff}} = \begin{cases} \sqrt{2\ln n} & n \geq 1.65 \\ 1 & \text{othewise} \end{cases} \tag{8.1-11}$$

$E[p]=\sqrt{2\ln n}$ in Eq. (8.1-11) is also plotted by a solid curve in Fig. 8.1-2 indicating the approximate behavior of $E[p]$.

It is observed from the above discussion that the mean vale and standard deviation of the peak factor p derived by Davenport (Eqs. (8.1-1a) and (8.1-1b)) are identical with those of Eq. (8.1-9a) with that

$$n_D=\frac{2L}{L_F}=\frac{1}{\sqrt{2\pi}}\frac{2L}{L_F^*}=\frac{1}{4.443}\frac{2L}{L_F^*} \tag{8.1-12}$$

The scaling factor $1/\sqrt{2\pi}$ in Eq. (8.1-12) is due to the fact that, for narrow band process, the peaks and troughs tend to appear twice within the apparent wave length L_F which is longer than the correlation distance L_F^* ($L_F=\sqrt{2\pi}L_F^*$) as shown in Fig. 8.1-1. However, the effect of n on $E[p]$ is not so sensitive that the difference is small between $E[p]$ with n given by Eqs. (8.1-7) and (8.1-12) as shown in Fig. 8.1-2. In fact, for example, for $n=40$ in Eq. (8.1-7), Eq. (8.1-12) gives $n_D=40/4.443=9.0$. From Fig. 8.1-2, the corresponding mean peak factors are read as $E[p]=2.9$ for $n=40$ and $E[p]=2.4$ for $n_D=9.0$ indicating little difference.

In turn, for a wide band process where L_F^* cannot define, the number $n=2L/L^*$ may tend to give smaller values than the true number of peak values. In fact, for pure wide band processes ($\varepsilon=1$) where the correlation function is expressed by the Dirac delta function, the correlation scale becomes a finite value of $2\pi S_{ff}(0)/\sigma_{ff}^2$. However, within this interval

$L^* = 2\pi S_{ff}(0) / \sigma_{ff}^2$, true peaks may tend to occur more than one. Hence, the mean value of the peak factor given by Eq. (8.1-10a) with $n = 2L / L^*$ may give a lower value of $E[p]$.

In conclusion, for practical use of the peak factor of the absolute maximum value of $f(x)$ over the range $0 \leq x \leq L$, as a conservative mean peak factor, Eq. (8.1-9a) may be appropriate with n given by Eq. (8.1-7), and Eq. (8.1-10a) with n given by Eq. (8.1-7) as a lower value of the mean peak factor. For more simplicity, Eq. (8.1-11) may be useful with n given by Eq. (8.1-7).

8.2 Seismic Ground RMS Estimate

In contrast to the earthquake resistant design of above ground structures where the inertial forces induced by ground acceleration are the main consideration, the spatial variation of the ground displacement is of primary importance for buried lifeline structures such as pipelines and tunnels. Consequently, the ground strains and relative displacements between two locations along pipelines play main roles in the seismic design of such buried lifeline structures.

Since the seismic ground motion displacement at a time instance along pipe axis (x) varies with location, it is expressed as $u(x)$. Therefore, the relative ground displacement $u_D(x)$ and the averaging ground strain $\varepsilon_D(x)$ between two points with separation distance D are given by

$$u_D(x) = u(x + D/2) - u(x - D/2)$$

$$\varepsilon_D(x) = \frac{1}{D} \int_{x-D/2}^{x+D/2} \varepsilon(y)dy = \frac{1}{D} u_D(x) \tag{8.2-1}$$

where the relationship $\varepsilon(x) = du(x)/dx$ between ground strain and ground displacement is used.

By comparing the averaging process and difference process in Eqs. (2.1) and (2.3) with Eq. (8.2-1), the following correspondences are clearly observed; $\varepsilon(x) \leftrightarrow f(x), u(x) \leftrightarrow F(x)$. From this analogy and Eq. (2.14), the deviations $\sigma_{\varepsilon_D}, \sigma_{u_D}$ of ground strain $\varepsilon_D(x)$ and the relative ground displacement $u_D(x)$ are given by

$$\sigma_{\varepsilon_D} = \frac{1}{D} \sigma_{u_D}$$

$$\sigma_{\varepsilon_D} = \frac{1}{D} \sqrt{2\left(R_{uu}(0) - R_{uu}(D)\right)} \tag{8.2-2a}$$

$$\sigma_{u_D} = \sqrt{2\left(R_{uu}(0) - R_{uu}(D)\right)}$$

Also, from the relationship between apparent wavelength and correlation distance of Eqs. (3.5b) and (3.8), the apparent wavelength L_u and the correlation scale (distance) L_u^* of $u(x)$ are given by

$$L_u = 2\pi \frac{\sigma_{uu}}{\sigma_{\varepsilon\varepsilon}}, \quad L_u^* = \frac{1}{\sqrt{2}\pi} L_u \tag{8.2-2b}$$

where σ_{uu} and $\sigma_{\varepsilon\varepsilon}$ are the deviations of ground displacement $u(x)$ and ground strain $\varepsilon(x)$.

Applying stochastic process theory, we can estimate the rms (root mean square) values σ_{u_D} and σ_{ε_D} of the relative displacements between two locations on a ground surface and the ground strain along the pipe axis at time instance from the following equations:

$$\frac{\sigma_{u_D}}{\sigma_{uu}} = \begin{cases} \sqrt{2}\dfrac{D}{L_u^*} & D \le L_u^* \\ \sqrt{2} & D > L_u^* \end{cases} \tag{8.2-3a}$$

$$\frac{\sigma_{\varepsilon_D}}{\sigma_{uu}} = \begin{cases} \dfrac{\sqrt{2}}{L_u^*} & D \le L_u^* \\ \dfrac{\sqrt{2}}{D} & D > L_u^* \end{cases} \tag{8.2-3b}$$

At now, two examples of exact solution of $\sigma_{u_D} / \sigma_{uu}$ are shown as follows:

Type 1 of Table 4.2-1: For a power spectrum of strain, we use

$$\begin{aligned} S_{\varepsilon\varepsilon}(\kappa) &= \frac{\sigma_{uu}^2}{2\cdot 0!} b\kappa^2 e^{-b|\kappa|} \\ \sigma_{\varepsilon\varepsilon}^2 &= \int_{-\infty}^{\infty} S_{\varepsilon\varepsilon}(\kappa)d\kappa = \frac{2!}{b^2}\sigma_{uu}^2 \end{aligned} \tag{8.2-4a}$$

Then,

$$\begin{aligned} S_{uu}(\kappa) &= \frac{S_{\varepsilon\varepsilon}(\kappa)}{\kappa^2} = \frac{\sigma_{uu}^2}{2\cdot 2!} be^{-b|\kappa|} \\ R_{uu}(\xi) &= \sigma_{uu}^2 \frac{b^2}{b^2+\xi^2} \end{aligned} \tag{8.2-4b}$$

The correlation scale of L_u^* is given by

$$L_u^* = \frac{1}{\sqrt{2\pi}}\left(2\pi\frac{\sigma_{uu}}{\sigma_{\varepsilon\varepsilon}}\right) = \frac{1}{\sqrt{2\pi}}\left(2\pi\frac{b}{\sqrt{2}}\right) = b \tag{8.2-4c}$$

As a result, the exact solution of $\sigma_{u_D} / \sigma_{uu}$ is given by

$$\frac{\sigma_{u_D}}{\sigma_{uu}} = \sqrt{2\left(1 - R_{uu}(D) / R_{uu}(0)\right)}$$
$$= \sqrt{\frac{2\left(D / L_u^*\right)^2}{1 + \left(D / L_u^*\right)^2}} \tag{8.2-4d}$$

Type 2 of Table 4.2-2: For a power spectrum of strain, we assume

$$S_{\varepsilon\varepsilon}(\kappa) = \frac{\sigma_{uu}^2}{2} \cdot \frac{b^3}{2\sqrt{\pi}} \kappa^4 \mathrm{e}^{-(b\kappa/2)^2}$$
$$\sigma_{\varepsilon\varepsilon}^2 = \int_{-\infty}^{\infty} S_{\varepsilon\varepsilon}(\kappa) d\kappa = \frac{6}{b^2} \sigma_{uu}^2 \tag{8.2-5a}$$

Then,

$$S_{uu}(\kappa) = \frac{S_{\varepsilon\varepsilon}(\kappa)}{\kappa^2} = \frac{\sigma_{uu}^2}{2} \cdot \frac{b^3}{2\sqrt{\pi}} \kappa^2 \mathrm{e}^{-(b\kappa/2)^2}$$
$$R_{uu}(\xi) = \sigma_{uu}^2 \left(1 - 2(\xi / b)^2\right) \mathrm{e}^{-(\xi/b)^2} \tag{8.2-5b}$$

The correlation scale of L_u^* is given by

$$L_u^* = \frac{1}{\sqrt{2\pi}} \left(2\pi \frac{\sigma_{uu}}{\sigma_{\varepsilon\varepsilon}}\right) = \frac{1}{\sqrt{2\pi}} \left(2\pi \frac{b}{\sqrt{6}}\right) = \frac{b}{\sqrt{3}} \tag{8.2-5c}$$

As a result, the exact solution of $\sigma_{u_D} / \sigma_{uu}$ is given by

$$\frac{\sigma_{u_D}}{\sigma_{uu}} = \sqrt{2\left(1 - R_{uu}(D) / R_{uu}(0)\right)}$$
$$= \sqrt{2\left(1 - \left(1 - \frac{2}{3}\left(D / L_u^*\right)^2\right)\right) \exp\left(-\frac{1}{3}\left(D / L_u^*\right)^2\right)} \tag{8.2-5d}$$

From Eq. (8.2-1), σ_{u_D} and σ_{ε_D} are related with the following equation.

$$\sigma_{u_D} = D\, \sigma_{\varepsilon_D} \tag{8.2-6a}$$

Hence,

$$\frac{\sigma_{u_D}}{\sigma_{uu}} = \frac{D\sigma_{\varepsilon_D}}{\sigma_{uu}} = \frac{D}{L_u^*}\frac{\sigma_{\varepsilon_D}}{(\sigma_{uu} / L_u^*)}$$

$$\log\frac{\sigma_{u_D}}{\sigma_{uu}} = \log\frac{D}{L_u^*} + \log\frac{\sigma_{\varepsilon_D}}{(\sigma_{uu} / L_u^*)} \qquad (8.2\text{-}6b)$$

The above logarithmic expression is useful for graphical representation of relationship among the values of $\sigma_{u_D} / \sigma_{uu}$, D / L_u^* and $\sigma_{\varepsilon_D} / (\sigma_{uu} / L_u^*)$ as shown in Fig. 8.2-1. The solid lines in Fig. 8.2-1 are the approximation given by Eqs. (8.2-3a) and (8.2-3b). The two dashed curves are the exact solutions of Eqs. (8.2-4d) and (8.2-5d) by using the type 1 of Table 4.2-1 and the type 2 of Table 4.2-2 for the power spectrum of ground strain.

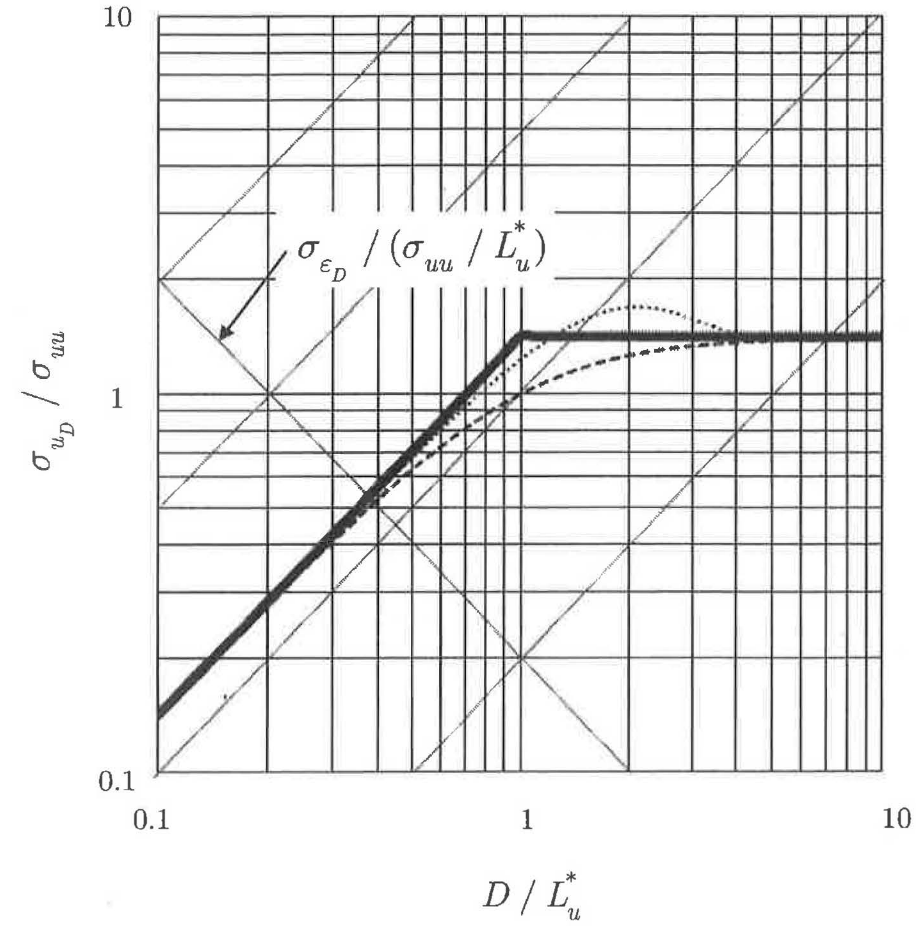


Fig. 8.2-1 $\sigma_{u_D} / \sigma_{uu}$, D / L_u^* and $\sigma_{\varepsilon_D} / (\sigma_{uu} / L_u^*)$ Diagram

To constructing Fig. 8.2-1, the parameters are only σ_{uu} of ground displacement, the correlation distance L_u^* and the relative distance D between two points on ground surface. More detail and field data analysis can be seen in the paper by Harada and Shinozuka (1986).

8.3 Miscellanea

In a digital time (spatial) series analysis and simulation, we must determine the upper cut-off frequency ω_u (upper cut-off wavenumber κ_u) above which the power spectral density function is considered to be zero. For this upper cut-off wavenumber κ_u, the spectral scales defined by Eq. (3.11) can be used as a measure of $\omega_u(\kappa_u)$ such that

$$\kappa_u \geq \kappa^* \quad \text{or} \quad \kappa_u \geq \kappa_F^* \left(\frac{2\pi}{L^*} \text{ or } \frac{2\pi}{L_F^*} \right) \tag{8.2-7}$$

In a stochastic finite element analysis where the material properties or boundary conditions are assumed to be stochastic, we face the determination of the finite element size corresponding to the randomness of the material properties in space. For this problem, the correlation scales may be useful as a measure of the relationships between the element size and the material randomness in space.

9. CONCLUSIONS

In this study, we reinterpret the correlation scales previously defined in the literature from the viewpoint of the statistical analysis of observed field data. By considering the averaging process and the difference process, two typical definitions for correlation scales are consistently derived, and also new definitions for the same correlation scales are obtained which make it possible to estimate the correlation scales from variances easily calculated from observed field data. The statistical assessments of the estimation of correlation scales from the variances are also briefly presented.

By extending the procedure for one dimensional stochastic process to two dimensional stochastic processes, the correlation scales (area) of two dimensional processes are defined and visually illustrated using a digital simulation technique. An estimation procedure for these correlation scales for two dimensional processes is presented.

Finally some new application examples of correlation scales defined and reinterpreted in this study are briefly presented. They are the applications of correlation scales into (1) the approximate distribution of the maximum values of stochastic processes over a finite length, (2) the estimation of the seismic ground rms (root mean square) strain, (3) a measure of the upper cut off frequency in the digital time series analysis and simulation, and (4) a measure of the relationships between the finite element size and the material randomness in stochastic finite element analysis.

10. ACKNOWLEDGEMENT

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