Harmonic Analysis And Simulation Of Homogeneous Stochastic Fields

HARMONIC ANALYSIS AND SIMULATION

OF

HOMOGENEOUS STOCHASTIC FIELDS

by

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and

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INTRODUCTION

In this paper, a spectral representation of stochastic fields is given in a form that is convenient for their simulation or digital generation of their sample functions. In a previous paper, Shinozuka and Jan (1972) discussed a simulation technique of multivariate multi-dimensional homogeneous as well as nonhomogeneous processes which represent frozen patterns of stochastic waves propagating in the direction specified by the wave number vector located in the first or last quadrant in an n-dimensional rectangular Cartesian coordinate system for wave numbers. The wave number vector is located in the first (last) quadrant if all the wave numbers are positive (negative). In this sense, the fields simulated by Shinozuka and Jan are not consistent with the general spectral representation of stochastic processes, although their simulated stochastic fields satisfy the target power spectral density (or correlation) functions. A revised version of the simulation technique was published by Shinozuka (1985) to satisfy this situation. The present paper provides a more detailed analysis in this direction.

The present paper also discusses how time-space stochastic processes, or stochastic waves, can be characterized within the framework of a second-order analysis. In this connection, numerical examples involving seismic array records in Taiwan (SMART-1) are worked out. Finally, a brief account is made in this paper as to how the spectral density functions of bi-variate two-dimensional stochastic fields or stochatic waves can be estimated from a set of data in a finite region. Although the present study restricts itself to bivariate two-dimensional cases for simplicity, the results may be easily extended to multi-variate multi-dimensional cases.

2. SPECTRAL REPRESENTATION AND SIMULATION OF BI-VARIATE ONE-DIMENSIONAL

STOCHASTIC FIELDS

2.1 Complex-Valued Stochastic Fields

In the harmonic analysis of stochastic fields, it is convenient to consider the fields to be complex-valued. Real-valued stochastic fields can be treated as a special case of complex-valued fields.

The complex stochastic fields f(x) and g(x) can be defined such that

$$\begin{aligned} f(x) &= f^{(1)}(x) + if^{(2)}(x) \\ g(x) &= g^{(1)}(x) + ig^{(2)}(x) \end{aligned} \tag{2.1}$$

where i is the imaginary unit and $f^{(1)}(x)$, $f^{(2)}(x)$, $g^{(1)}(x)$ and $g^{(2)}(x)$ are real stochastic fields as functions of the space coordinate x. The expected values can be defined as

$$\begin{aligned} E[f(x)] &= E[f^{(1)}(x)] + iE[f^{(2)}(x)] \\ E[g(x)] &= E[g^{(1)}(x)] + iE[g^{(2)}(x)] \end{aligned} \tag{2.2}$$

where E[·] is the expectation operator.

If the fields are homogeneous with zero mean;

$$E[f^{(1)}(x)] = E[f^{(2)}(x)] = E[g^{(1)}(x)] = E[g^{(2)}(x)] = 0 \tag{2.3}$$

Then, the covariance function matrix can be defined as

$$R(\xi) = \begin{pmatrix} R_{ff}(\xi) & R_{fg}(\xi) \\ R_{gf}(\xi) & R_{gg}(\xi) \end{pmatrix} = \begin{pmatrix} E[f(x+\xi)\overline{f}(x)] & E[f(x+\xi)\overline{g}(x)] \\ E[g(x+\xi)\overline{f}(x)] & E[g(x+\xi)\overline{g}(x)] \end{pmatrix} \tag{2.4}$$

where $\overline{f}(x)$ denotes the complex conjugate of f(x); $\overline{f}(x) = f^{(1)}(x) - if^{(2)}(x)$. From Eq. 2.4, it can be shown that the covariance function satisfies the following condition:

$$R_{jk}(\xi) = \bar{R}_{kj}(-\xi) \tag{2.5}$$

where j denotes f or g and so does k; this notation will be used throughout.

The stochastic fields f(x) and g(x) can be represented by the following integrals:

$$f(x) = \int_{-\infty}^{\infty} e^{i\kappa x} dZ_f(\kappa) \qquad\qquad g(x) = \int_{-\infty}^{\infty} e^{i\kappa x} dZ_g(\kappa) \tag{2.6}$$

where $dZ_f(\kappa)$ and $dZ_g(\kappa)$ are the orthogonal increments satisfying the following conditions:

$$dZ_j(\kappa) = Z_j(\kappa+d\kappa) - Z_j(\kappa) \tag{2.7}$$

$$E[dZ_j(\kappa)] = 0 \tag{2.8}$$

$$E\left[dZ_j(\kappa)\overline{dZ}_k(\kappa)\right] = E\left[\{Z_j(\kappa+\Delta\kappa) - Z_j(\kappa)\}\overline{\{Z_k(\kappa'+\Delta\kappa') - Z_k(\kappa)\}}\right] = dF_{jk}(\kappa) \tag{2.9}$$

and

$$E\left[dZ_j(\kappa)\overline{dZ}_k(\kappa')\right] = E\left[\{Z_j(\kappa+\Delta\kappa) - Z_j(\kappa)\}\overline{\{Z_k(\kappa'+\Delta\kappa') - Z_k(\kappa')\}}\right] = 0 \tag{2.10}$$

if $(\kappa,\kappa+\Delta\kappa)$ and $(\kappa',\kappa'+\Delta\kappa')$ are disjoint. For a differentiable $F_{jk}(\kappa)$, Eq. 2.9 becomes

$$E\left[dZ_j(\kappa)\overline{dZ}_k(\kappa)\right] = S_{jk}(\kappa)d\kappa \tag{2.11}$$

where

$$S_{jk}(\kappa) = \frac{dF_{jk}(\kappa)}{d\kappa} \tag{2.12}$$

Substitution of Eq. 2.6 into Eq. 2.4 and use of Eqs. 2.9 and 2.10 result in

$$R_{jk}(\xi) = \int_{-\infty}^{\infty} e^{ik\xi} dF_{jk}(\kappa) \tag{2.13}$$

If $F_{jk}(\kappa)$ is differentiable, then the integral of Eq. 2.13 reduces to:

$$R_{jk}(\xi) = \int_{-\infty}^{\infty} e^{i\kappa\xi} S_{jk}(\kappa) d\kappa \tag{2.14}$$

The inverse transform gives $S_{jk}(\kappa)$ in terms of $R_{jk}(\xi)$:

$$S_{jk}(\kappa) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\kappa\xi} R_{jk}(\xi) d\xi \tag{2.15}$$

Equations 2.14 and 2.15 represent the well-known Wiener-Khintchine transform pair.

2.2 Real-Valued Stochastic Fields

Consider that the complex-valued functions $F_{jk}(\kappa)$ and $Z_j(\kappa)$ introduced above are represented in terms of orthogonal increments such that

$$dF_{jk}(\kappa) = \frac{1}{2}[dF_{jk}^{(1)}(\kappa) - idF_{jk}^{(2)}(\kappa)] \tag{2.16}$$

$$dZ_j(\kappa) = \frac{1}{2}[dU_j^{(1)}(\kappa) - idU_j^{(2)}(\kappa)] \tag{2.17}$$

where $dF_{jk}^{(1)}$, $dF_{jk}^{(2)}$, $dU_j^{(1)}$ and $dU_j^{(2)}$ are real-valued. Substitution of Eqs. 2.16 and 2.17 respectively into Eqs. 2.6 and 2.7 yields the following alternative expressions for $R_{jk}(\xi)$ and $j(x)$:

$$R_{jk}(\xi) = \frac{1}{2} \int_{-\infty}^{\infty} [\cos \kappa\xi dF_{jk}^{(1)}(\kappa) + \sin \kappa\xi dF_{jk}^{(2)}(\kappa)]$$

$$+ i \frac{1}{2} \int_{-\infty}^{\infty} [\sin \kappa\xi dF_{jk}^{(1)}(\kappa) - \cos \kappa\xi dF_{jk}^{(2)}(\kappa)] \tag{2.18}$$

and

$$j(x) = \frac{1}{2}\int_{-\infty}^{\infty} \left[\cos \kappa x dU_j^{(1)}(\kappa) + \sin \kappa x dU_j^{(2)}(\kappa)\right]$$

$$+ i \frac{1}{2}\int_{-\infty}^{\infty} \left[\sin \kappa x dU_j^{(1)}(\kappa) - \cos \kappa x dU_j^{(2)}(\kappa)\right] \quad (2.19)$$

For real-valued stochastic fields, the imaginary parts of $R_{jk}(\xi)$ and $j(x)$ in the previous equations must be zero. This requires that $dF_{jk}(\kappa)$ and $dZ_j(\kappa)$ satisfy:

$$dF_{jk}(-\kappa) = \overline{dF}_{jk}(\kappa) \quad (2.20)$$

and

$$dZ_j(-\kappa) = \overline{dZ}_j(\kappa) \quad (2.21)$$

Equations 2.20 and 2.21 imply that the real parts of $dF_{jk}(\kappa)$ and $dZ_j(\kappa)$ are even functions of κ, while their imaginary parts are odd functions of κ. Because of this, Eqs. 2.18 and 2.19 reduce to

$$R_{jk}(\xi) = R_{jk}^{(1)}(\xi) = \int_0^{\infty} \left[\cos \kappa\xi dF_{jk}^{(1)}(\kappa) + \sin \kappa\xi dF_{jk}^{(2)}(\kappa)\right] \quad (2.22)$$

and

$$j(x) = j^{(1)}(x) = \int_0^{\infty} \left[\cos \kappa x dU_j^{(1)}(\kappa) + \sin \kappa x dU_j^{(2)}(\kappa)\right] \quad (2.23)$$

The increments $dU_j^{(1)}(\kappa)$ and $dU_j^{(2)}(\kappa)$ must first satisfy the following conditions:

$$E\left[dU_j^{(n)}(\kappa) dU_k^{(m)}(\kappa')\right] = 0 \quad (2.24)$$

for any combinations of n, m, j and k, while

$$E\left[dU_j^{(1)}(\kappa)d\bar{U}_j^{(2)}(\kappa)\right] = 0 \tag{2.25}$$

and

$$E\left[dU_j^{(1)}(\kappa)dU_k^{(1)}(\kappa)\right] = E\left[dU_j^{(2)}(\kappa)dU_k^{(2)}(\kappa)\right] = dF_{jk}^{(1)}(\kappa) \tag{2.26}$$

$$E\left[dU_j^{(2)}(\kappa)dU_k^{(1)}(\kappa)\right] = - E\left[dU_j^{(1)}(\kappa)dU_k^{(2)}(\kappa)\right] = dF_{jk}^{(2)}(\kappa) \tag{2.27}$$

Equations 2.26 and 2.27, respectively, are derived from Eqs. 2.17 and 2.19. It is not hard to show Eq. 2.25 using Eqs. 2.26–2.30. In fact,

$$\begin{aligned}
R_{jk}^{(1)}(\xi) &= E[j^{(1)}(x+\xi)k^{(1)}(x)] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[\cos\{\kappa(x+\xi)\}dU_j^{(1)}(\kappa) + \sin\{\kappa(x+\xi)\}dU_j^{(2)}(\kappa)\right] \\
&\quad \times \left[\cos\kappa' x dU_k^{(1)}(\kappa') + \sin \kappa' x dU_k^{(2)}(\kappa')\right] \\
&= \int_0^{\infty} \left[\cos\{\kappa(x+\xi)\}\cos \kappa x \; E\left[dU_j^{(1)}(\kappa)dU_k^{(1)}(\kappa)\right]\right. \\
&\quad + \cos\{\kappa(x+\xi)\}\sin \kappa x \; E\left[dU_j^{(1)}(\kappa)dU_k^{(2)}(\kappa)\right] \\
&\quad + \sin\}\kappa(x+\xi)\}\cos \kappa x \; E\left[dU_j^{(2)}(\kappa)dU_k^{(1)}(\kappa)\right] \\
&\quad \left. + \sin\{\kappa(x+\xi)\}\sin \kappa x \; E\left[dU_j^{(2)}(\kappa)dU_k^{(2)}(\kappa)\right]\right] \\
&= \int_0^{\infty} \left[\cos \kappa\xi dF_{jk}^{(1)}(\kappa) + \sin x\xi dF_{jk}^{(2)}(\kappa)\right]
\end{aligned} \tag{2.31}$$

Especially when j = k, then from Eqs. 2.5, 2.28 and 2.30,

$$R_{jj}^{(1)}(\xi) = R_{jj}^{(1)}(-\xi) , \qquad dF_{jj}^{(2)}(\kappa) = 0 \tag{2.32}$$

Hence, Eq. 2.31 reduces to

$$R_{jj}^{(1)}(\xi) = \int_0^{\infty} \cos \kappa\xi dF_{jj}^{(1)}(\kappa) \tag{2.33}$$

If $dF_{jj}^{(1)}(\kappa)$ is continuous, the power spectral density function $S_{jj}(\kappa)$ can be defined as in Eq. 2.12. Then, Eq. 2.33 becomes such that

$$R_{jj}^{(1)}(\xi) = 2 \int_0^\infty \cos \kappa\xi S_{jj}(\kappa) d\kappa \tag{2.34}$$

The inverse transform gives $S_{jj}(\kappa)$ as

$$S_{jj}(\kappa) = \frac{1}{\pi} \int_0^\infty \cos \kappa\xi R_{jj}^{(1)}(\xi) d\xi \tag{2.35}$$

FromEq. 2.35, it is observed that the power spectral density function $S_{jj}(\kappa)$ is symmetrical with respect to κ. Hence Eqs. 2.34 and 2.35 are also expressed as

$$R_{jj}^{(1)}(\xi) = \int_{-\infty}^\infty \cos \kappa\xi S_{jj}(\kappa) d\kappa \tag{2.36a}$$

$$S_{jj}(\kappa) = \frac{1}{2\pi} \int_{-\infty}^\infty \cos \kappa\xi R_{jj}^{(1)}(\xi) d\xi \tag{2.36b}$$

2.3 Simulation Method

We consider the simulation method of the homogeneous stochastic fields $f(x)$ and $g(x)$ under the condition that the power spectral density function $S_{jk}(\kappa)$ is specified such that

$$\begin{bmatrix} S_{ff}(\kappa) & S_{fg}(\kappa) \\ S_{gf}(\kappa) & S_{gg}(\kappa) \end{bmatrix} \tag{2.37}$$

$$S_{gf}(\kappa) = \bar{S}_{fg}(\kappa)$$

Since the power spectral density function consitutes the Hermitian and non-negative definite matrix (Yaglowm, 1962), Eq. 2.37 can be decomposed as

$$\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ 0 & a_{22} \end{bmatrix} = \begin{bmatrix} |a_{11}|^2, & a_{11}a_{21} \\ a_{21}a_{11}, & |a_{21}|^2 + |a_{22}|^2 \end{bmatrix} \tag{2.38}$$

where a_{jk} can be obtained by equating Eq. 2.37 with Eq. 2.38 such that

$$a_{11} = \sqrt{S_{ff}(\kappa)}\; e^{i\psi_2(\kappa)} = |a_{11}| e^{i\psi_2(\kappa)}$$

$$a_{21} = \frac{|S_{fg}(\kappa)|}{\sqrt{S_{ff}(\kappa)}}\; e^{i\{\psi_2(\kappa)+\alpha_{21}(\kappa)\}} = |a_{21}| e^{i\{\psi_2(\kappa) + \alpha_{21}(\kappa)\}}$$

$$a_{22} = \sqrt{\frac{S_{ff}(\kappa)S_{gg}(\kappa) - |S_{fg}(\kappa)|^2}{S_{ff}(\kappa)}}\; e^{i\psi_2(\kappa)} = |a_{22}| e^{i\psi_2(\kappa)} \tag{2.39}$$

where $\psi_1(\kappa)$ and $\psi_2(\kappa)$ are arbitrary phase angles and

$$\alpha_{21}(\kappa) = \tan^{-1}\left\{ \frac{Im[S_{fg}(\kappa)]}{Re[S_{fg}(\kappa)]} \right\} \tag{2.40}$$

On the other hand, the covariance of the orthogonal increment $dZ_j(\kappa)$ is given by Eq. 2.17b, that is,

$$\begin{aligned} E[dZ_f\, \overline{dZ}_f] &= S_{ff}(\kappa)d\kappa \\ E[dZ_f\, \overline{dZ}_g] &= S_{fg}(\kappa)d\kappa \\ E[dZ_g\, \overline{dZ}_g] &= S_{gg}(\kappa)d\kappa \end{aligned} \tag{2.41}$$

Comparison of Eqs. 2.38 and 2.41 motivates the introduction of a new definition for the orthogonal increments dZ_f and dZ_g to efficiently express the orthogonal increments in terms of the power spectral density functions as follows:

$$dZ_f = dZ_{ff} \tag{2.42}$$

$$dZ_g = dZ_{gf} + dZ_{gg} \tag{2.43}$$

where

$$E[dZ_{ff}\, \overline{dZ}_{gg}] = E[dZ_{gf}\, \overline{dZ}_{gg}] = 0 \tag{2.44}$$

and similarly

$$dZ_f = \frac{1}{2}\left[dU_{ff}^{(1)} - i dU_{ff}^{(2)}\right] \tag{2.45}$$

$$dZ_g = \frac{1}{2}\left[dU_{gf}^{(1)} - i dU_{gf}^{(2)}\right] + \frac{1}{2}\left[dU_{gg}^{(1)} - i dU_{gg}^{(2)}\right] \tag{2.46}$$

where $dU_{jk}^{(1)}$ and $dU_{jk}^{(2)}$ are real-valued.

Substitution of Eqs. 2.42 and 2.43 into Eq. 2.41 and taking into account Eq. 2.44 yields Eq. 2.47.

$$\begin{bmatrix} E[dZ_f\,\overline{dZ}_f], & E[dZ_f\,\overline{dZ}_g] \\ E[dZ_g\,\overline{dZ}_f], & E[dZ_g\,\overline{dZ}_g] \end{bmatrix} = \begin{bmatrix} E[|dZ_{ff}|^2], & E[dZ_{ff}\,\overline{dZ}_{gf}] \\ E[dZ_{gfr}\,\overline{dZ}_{ff}], & E[|dZ_{gf}|^2 + |dZ_{gg}|^2] \end{bmatrix} \tag{2.47}$$

Comparing Eq. 2.38 with Eq. 2.47, the orthogonal increments can be obtained such that

$$\begin{aligned} dZ_{ff}(\kappa) &= |a_{11}|\sqrt{d\kappa}\; e^{i\psi_1(\kappa)} \\ dZ_{gf}(\kappa) &= |a_{21}|\sqrt{d\kappa}\; e^{i\{\psi_1(\kappa)+\alpha_{21}(\kappa)\}} \\ dZ_{gg}(\kappa) &= |a_{22}|\sqrt{d\kappa}\; e^{i\psi_2(\kappa)} \end{aligned} \tag{2.48}$$

also, taking into account Eqs. 2.45 and 2.46,

$$dU_{ff}^{(1)}(\kappa) = 2|a_{11}|\sqrt{d\kappa}\,\cos\psi_1(\kappa) \tag{2.49a}$$

$$dU_{ff}^{(2)}(\kappa) = -\,2|a_{11}|\sqrt{d\kappa}\,\sin\psi_1(\kappa) \tag{2.49b}$$

$$dU_{gf}^{(1)}(\kappa) = 2|a_{21}|\sqrt{d\kappa}\,\cos\{\psi_1(\kappa) + \alpha_{21}(\kappa)\} \tag{2.49c}$$

$$dU_{gf}^{(2)}(\kappa) = -\,2|a_{21}|\sqrt{d\kappa}\,\sin\{\psi_1(\kappa) + \alpha_{21}(\kappa)\} \tag{2.49d}$$

$$dU_{gg}^{(1)}(\kappa) = 2|a_{22}|\sqrt{d\kappa}\,\cos\psi_2(\kappa) \tag{2.49e}$$

$$dU_{gg}^{(2)}(\kappa) = -\,2|a_{22}|\sqrt{d\kappa}\,\sin\psi_2(\kappa) \tag{2.49f}$$

In Eqs. 2.48 and 2.49, the arbitrary phase angles $\psi_1(\kappa)$ and $\psi_2(\kappa)$ must be appropriate random functions so that the orthogonal increment $dZ_j(\kappa)$ satisfy the orthogonal conditions as given in Eqs. 2.16 and 2.17. If we choose independent random phases uniformly distributed between 0 and 2π for $\psi_1(\kappa)$ and $\psi_2(\kappa)$, it is easy to show that Eqs. 2.48 and 2.49 satisfy Eqs. 2.16 and 2.17, respectively.

From Eqs. 2.26, 2.45, 2.46 and 2.49, the real-valued stochastic fields $f^{(1)}(x)$ and $g^{(1)}(x)$ can be expressed as

$$f^{(1)}(x) = \int_0^\infty 2|a_{11}(\kappa)|\sqrt{d\kappa}\ \cos\{\kappa x + \psi_1(\kappa)\} \tag{2.50a}$$

and

$$g^{(1)}(x) = \int_0^\infty 2|a_{21}(\kappa)|\sqrt{d\kappa}\ \cos\{\kappa x + \psi_1(\kappa) + \alpha_{21}(\kappa)\}$$
$$+ \int_0^\infty 2|a_{22}(\kappa)|\sqrt{d\kappa}\ \cos\{\kappa x + \psi_2(\kappa)\} \tag{2.50b}$$

The integrals mean

$$f^{(1)}(x) = \lim_{\Delta\kappa\to 0} \sum_{n=1}^{N} 2|a_{11}(\kappa_n)|\sqrt{\Delta\kappa}\ \cos\{\kappa_n x + \psi_{1_n}\} \tag{2.51a}$$

and

$$g^{(1)}(x) = \lim_{\Delta\kappa\to 0} \sum_{n=1}^{N} 2|a_{21}(\kappa_n)|\sqrt{\Delta\kappa}\ \cos\{\kappa_n x + \psi_{1_n} + \alpha_{21}(\kappa_{x_n})\}$$
$$+ 2|a_{22}(\kappa_n)|\sqrt{\Delta\kappa}\ \cos\{\kappa_n x + \psi_{2_n}\} \tag{2.51b}$$

Equation 2.50 is identical to that used by Shinozuka and Jan (1972). As shown by them, making use of the FFT (Fast Fourier Transform) technique in the summations appearing in Eq. 2.50 drastically reduces the computing time.

3. SPECTRAL REPRESENTATION OF BI-VARIATE TWO-DIMENSIONAL STOCHASTIC FIELDS

3.1 Complex-Values Stochastic Fields

The previous procedure described in Section 2 can be directly used for the bi-variate two-dimensional case. Almot all the equations in Sections 3 and 4 are similar, but the equations for real-valued fields are quite different. This difference is also quite important in the simulation of real-valued stochastic fields as explained in the numerical examples (see Section 6). To explain the difference, a similar procedure and equations are provided.

The complex stochastic fields f(x,y) and g(x,y) can be defined as

$$f(x,y) = f^{(1)}(x,y) + if^{(2)}(x,y) \tag{3.1}$$

$$g(x,y) = g^{(1)}(x,y) + ig^{(2)}(x,y) \tag{3.2}$$

where $i = \sqrt{-1}$, $f^{(1)}(x,y)$, $f^{(2)}(x,y)$, $g^{(1)}(x,y)$ and $g^{(2)}(x,y)$ are real stochastic fields, and x,y denote real coordinates. The mean can be defined as

$$E[f(x,y)] = E[f^{(1)}(x,y)] + iE[f^{(2)}(x,y)] \tag{3.3}$$

$$E[g(x,y)] = E[g^{(1)}(x,y)] + iE[g^{(2)}(x,y)] \tag{3.4}$$

where E[•] is the expectation operator.

Now suppose that the fields are homogeneous stochastic fields with zero-mean. Then, the covariance function of the fields can be defined in matrix form such that

$$\underline{R}(\xi_x,\xi_y) = \begin{bmatrix} R_{ff}(\xi_x,\xi_y), & R_{fg}(\xi_x,\xi_y) \\ R_{gf}(\xi_x,\xi_y), & R_{gg}(\xi_x,\xi_y) \end{bmatrix} = \begin{bmatrix} E[f(x+\xi_x,y+\xi_y)\bar{f}(x,y)], & E[f(x+\xi_x,y+\xi_y)\bar{g}(x,y)] \\ E[g(x+\xi_x,y+\xi_y)\bar{f}(x,y)], & E[g(x+\xi_x,y+\xi_y)\bar{g}(x,y)] \end{bmatrix} \tag{3.5}$$

where $\bar{f}(x,y)$ denotes the complex conjugate of f(x,y), i.e., $\bar{f}(x,y) = f^{(1)}(x,y)$

$- if^{(2)}(x,y)$. Thus

$$R_{jk}(\xi_x,\xi_y) = \overline{R}_{kj}(-\xi_2,-\xi_y) \qquad j,k = f,g \qquad (3.6)$$

that is, the covariance matrix for bi-variate stochastic fields constitutes Hermitian. In particular, the variances of the diagonal term are real and positive, that is,

$$Var[f] = R_{ff}(0) = E[f(x,y)\overline{f}(x,y)] = E[\{f^{(1)}(x,y)\}^2 + \{f^{(2)}(x,y)\}^2] \qquad (3.7)$$

$$Var[g] = R_{gg}(0) = E[g(x,y)\overline{g}(x,y)] = E[\{g^{(1)}(x,y)\}^2 + \{g^{(2)}(x,y)\}^2] \qquad (3.8)$$

It can be shown for the homogeneous stochastic fields f(x,y) and g(x,y), that the covariance function $R_{jk}(\xi_x,\xi_y)$ (j,k = f,g) can always be represented as follows (Yaglom, 1962):

$$R_{jk}(\xi_x,\xi_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\kappa_x\xi_x+\kappa_y\xi_y)} dF_{jk}(\kappa_x,\kappa_y) \qquad (3.9)$$

and the fields themselves can be represented as

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\kappa_x x+\kappa_y y)} dZ_f(\kappa_x,\kappa_y) \qquad (3.10)$$

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\kappa_x x+\kappa_y y)} dZ_g(\kappa_x,\kappa_y) \qquad (3.11)$$

where the integral means the Fourier-Stieljes integral standing for the limit, for instance, for the integral of Eq. 1.10,

$$f(x,y) = \lim_{\substack{a\to\infty \\ b\to\infty}} \int_{-a}^{a}\int_{-b}^{b} e^{i(\kappa_x x+\kappa_y y)} dZ_j(\kappa_x,\kappa_y)$$

$$= \lim_{\substack{a\to\infty \\ b\to\infty}}[\lim_{\substack{\Delta\kappa_x\to 0 \\ \Delta\kappa_y\to 0}} \sum_{n=-N}^{N} \sum_{m=-M}^{M} e^{i(\kappa_{x_m} x+\kappa_{y_m} y)} \{Z_f(\kappa_{x_n}+\Delta\kappa_x,\kappa_{y_m}+\Delta\kappa_y) \qquad (3.12)$$

$$- Z_f(\kappa_{x_n}+\Delta\kappa_x,\kappa_{y_m}) - Z_f(\kappa_{x_n},\kappa_{y_m}+\Delta\kappa_y) + Z_f(\kappa_{x_n},\kappa_{y_m})]]$$

wehre the summation is over all the subjects appearing in the partition (see Fig. 1.1)

$$- a = \kappa_{x_{-N}} < \ldots \kappa_{x_n} \ldots < \kappa_{x_N} = a$$

$$- b = \kappa_{y_{-M}} < \ldots \kappa_{y_m} \ldots < \kappa_{y_M} = b \tag{3.13}$$

$$\Delta\kappa_x = \kappa_{x_m} - \kappa_{x_{m-1}}, \qquad \Delta\kappa_y = \kappa_{y_m} - \kappa_{y_{m-1}}$$

The function $F_{jk}(\kappa_x,\kappa_y)$ which is a non-negatie and non-decreasing function is called the spectral function of the fields $f(x,y)$ and $g(x,y)$. When the function $F_{jk}(\kappa_x,\kappa_y)$ is continuous, its derivative is called the spectral density function $S_{jk}(\kappa_x,\kappa_y)$, that is

$$S_{jk}(\kappa_x,\kappa_y) = \frac{\partial^2 F_{jk}(\kappa_x,\kappa_y)}{\partial\kappa_x\partial\kappa_y} \tag{3.14}$$

Then, the Fourier-Stieltjes integral of Eq. 3.9 reduces to the Fourier interval as

$$R_{jk}(\xi_x,\xi_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\kappa_x\xi_x+\kappa_y\xi_y)} S_{jk}(\kappa_x,\kappa_y)d\kappa_x d\kappa_y \tag{3.15}$$

The inverse transformation yields the spectral density function as

$$S_{jk}(\kappa_x,\kappa_y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\kappa_x\xi_x+\kappa_y\xi_y)} R_{jk}(\xi_x,\xi_y)d\xi_x d\xi_y \tag{3.16}$$

Equations 3.15 and 3.16 are the well-known Wiener-Khintchine relationships.

Turning to the representations of stochastic fields given by Eqs. 3.10

and 3.11, these representations imply that $f(x,y)$ and $g(x,y)$ can be written as the sum of many elementary waves $\exp[i(\kappa_x x+\kappa_y y)]$ with complex orthogonal random amplitudes $dZ_f(\kappa_x,\kappa_y)$ and $dZ_g(\kappa_x,\kappa_y)$, respectively. The orthogonal random amplitude is generally called the orthogonal increment which is defined as follows and satisfies the following conditions:

$$dZ_j(\kappa_x,\kappa_y) = Z_j(\kappa_x+d\kappa_x,\kappa_y+d\kappa_y) - Z_j(\kappa_x+d\kappa_x,\kappa_y) - Z_j(\kappa_x,\kappa_y+d\kappa_y) + Z_j(\kappa_x,\kappa_y) \tag{3.17}$$

and, if the regions $(\kappa_x,.\kappa_y)$ and (κ_x',κ_y') are disjointed,

$$E\left[dZ_j(\kappa_x,\kappa_y)\overline{dZ_k}(\kappa_x',\kappa_y')\right] = 0 \tag{3.18}$$

The covariance of $dZ_j(\kappa_x,\kappa_y)$ and $dZ_k(\kappa_x,\kappa_y)$ is

$$E\left[dZ_j(\kappa_x,\kappa_y)\overline{dZ_k}(\kappa_x,\kappa_y)\right] = dF_{jk}(\kappa_x,\kappa_y) \tag{3.19a}$$

For continuous $F_{jk}(\kappa_x,\kappa_y)$,

$$E\left[dZ_j(\kappa_x,\kappa_y)\overline{dZ_k}(\kappa_x,\kappa_y)\right) = S_{jk}(\kappa_x,\kappa_y)d\kappa_x d\kappa_y \tag{3.19b}$$

Due to Eqs. 3.18 and 3.19 (orthogonal conditions), it is easily confirmed that the covariance function is given by Eq. 3.9. In fact,

$$R_{jk}(\xi_x,\xi_y) = E\left[j(x+\xi_x,y+\xi_y)\overline{k}(x,y)\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\{\kappa_x(x+\xi_x)+\kappa_y(y+\xi_y)\}} e^{-i\{\kappa_x' x+\kappa_y' y\}}$$

$$\times E\left[dZ_j(\kappa_x,\kappa_y)\overline{dZ_k}(\kappa_x',\kappa_y')\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\kappa_x\xi_x+\kappa_y\xi_y)}\, dF_{jk}(\kappa_x,\kappa_y) \tag{3.20}$$

In deriving the last element of Eq. 3.20, Eqs. 3.18 and 3.19 have been used.

3.2 Real-Valued Stochastic Fields

In this section, we consider real-valued stochastic fields using the results described in Section 1.1. Suppose the complex-valued spectral distribution function $F_{jk}(\kappa_x,\kappa_y)$ and orthogonal function $Z_j(\kappa_x,\kappa_y)$ are represented in terms of the increment such that

$$dF_{jk}(\kappa_x,\kappa_y) = \frac{1}{2}\left[dF_{jk}^{(1)}(\kappa_x,\kappa_y) - i\,dF_{jk}^{(2)}(\kappa_x,\kappa_y)\right] \qquad (3.21)$$

$$dZ_j(\kappa_x,\kappa_y) = \frac{1}{2}\left[dU_j^{(1)}(\kappa_x,\kappa_y) - i\,dU_j^{(2)}(\kappa_x,\kappa_y)\right] \qquad (3.22)$$

where $dF_{jk}^{(1)}$ and $dF_{jk}^{(2)}$ are real-valued spectral distribution functions associated with the real and imaginary parts of dF_{jk}. The functions $dU_j^{(1)}$ and $dU_j^{(2)}$ are also real-valued increments associated with the real and imaginary parts of dZ_j.

Substitution of Eqs. 3.21 and 3.22 into Eqs. 3.9-3.11 yields alternative expressions for $R_{jk}(\xi_x,\xi_y)$ and $j(x,y)$ such as

$$R_{jk}(\xi_x,\xi_y) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\cos(\kappa_x\xi_x+\kappa_y\xi_y)dF_{jk}^{(1)}(\kappa_x,\kappa_y) + \sin(\kappa_x\xi_x+\kappa_y\xi_y)dF_{jk}^{(2)}(\kappa_x,\kappa_y)\right]$$
$$+ i\,\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\sin(\kappa_x\xi_x+\kappa_y\xi_y)dF_{jk}^{(1)}(\kappa_x,\kappa_y) - \cos(\kappa_x\xi_x+\kappa_y\xi_y)dF_{jk}^{(2)}(\kappa_x,\kappa_y)\right] \qquad (3.23)$$

and the fields $f(x,y)$ and $g(x,y)$ are for $j = f,g$,

$$j(x,y) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\cos(\kappa_x x + \kappa_y y)dU_j^{(1)}(\kappa_x,\kappa_y) + \sin(\kappa_x x + \kappa_y y)dU_j^{(2)}(\kappa_x,\kappa_y)\right]$$

$$+ i\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\sin(\kappa_x x+\kappa_y y)dU_j^{(1)}(\kappa_x,\kappa_y) - \cos(\kappa_x x+\kappa_y y)dU_j^{(2)}(\kappa_x,\kappa_y)] \tag{3.24}$$

For real-valued stochastic fields, the imaginary parts of $R_{jk}(\xi_x,\xi_y)$ and $j(x,y)$ given by Eqs. 3.23 and 3.24 must be zero. This condition requires the following relationships for $dF_{jk}(\kappa_x,\kappa_y)$ and $dZ_j(\kappa_x,\kappa_y)$:

$$dF_{jk}(-\kappa_x,-\kappa_y) = \overline{dF}_{jk}(\kappa_x,\kappa_y) \tag{3.25}$$

$$dF_{jk}(-\kappa_x,\kappa_y) = \overline{dF}_{jk}(\kappa_x,-\kappa_y) \tag{3.26}$$

and

$$dZ_j(-\kappa_x,-\kappa_y) = \overline{dZ}(\kappa_x,\kappa_y) \tag{3.27}$$

$$dZ_j(-\kappa_x,\kappa_y) = \overline{dZ}(\kappa_x,-\kappa_y) \tag{3.28}$$

Equations 3.25 and 3.26 imply that the bi-spectral distribution functions $F_{jk}(\kappa_x,\kappa_y)$ and $F_{jk}(\kappa_x,-\kappa_y)$ are necessary for representing the covariance function $R_{jk}^{(1)}(\xi_x,\xi_y)$ of the real-valued homogeneous stochastic fields $j^{(1)}(x,y)$ and $k^{(1)}(x,y)$ (see Fig. 2). Equations 3.27 and 3.28 also imply that the two orthogonal increments $dZ_j(\kappa_x,\kappa_y)$ and $dZ_j(\kappa_x,\kappa_y)$ are needed for the spectral representation of the real-valued stochastic field $j^{(1)}(x,y)$ (see Fig. 2a). If $F_{jk}(\kappa_x,\kappa_y) = F_{jk}(\kappa_x-\kappa_y)$ or $Z_j(\kappa_x,\kappa_y) = Z_j(\kappa_x-\kappa_y)$, the stochastic field is called quadrant symmetry (see Fig. 2b).

Substitution of the conditions given by Eqs. 3.25-3.28 into Eqs. 3.9-3.11 or Eqs. 3.23 and 3.24 yields the required fundamental expressions for real-valued two-dimensional stochastic fields such that

$$R_{jk}(\xi_x,\xi_y) = R_{jk}^{(1)}(\xi_x,\xi_y)$$

$$= \int_0^{\infty}\int_0^{\infty}[\cos(\kappa_x\xi_x+\kappa_y\xi_y)dF_{jk}^{(1)}(\kappa_x,\kappa_y) + \sin(\kappa_x\xi_x+\kappa_y\xi_y)dF_{jk}^{(2)}(\kappa_x,\kappa_y)]$$

$$= \int_0^{\infty}\int_0^{\infty}[\cos(\kappa_x\xi_x-\kappa_y\xi_y)dF_{jk}^{(1)}(\kappa_x,-\kappa_y- + \sin(\kappa_x\xi_x-\kappa_y\xi_y)dF_{jk}^{(2)}(\kappa_x,-\kappa_y)]$$

(3.29)

where j = f,g and the field itself is

$$j(x,y) = j^{(1)}(x,y)$$

$$= \int_0^\infty\int_0^\infty \left[\cos(\kappa_x x+\kappa_y y)dU_j^{(1)}(\kappa_x,\kappa_y) + \sin(\kappa_x x+\kappa_y y)dU_j^{(2)}(\kappa_x,\kappa_y)\right]$$

$$+ \int_0^\infty\int_0^\infty \left[\cos(\kappa_x x-\kappa_y y)dU_j^{(1)}(\kappa_x,-\kappa_y) + \sin(\kappa_x x-\kappa_y y)dU_j^{(2)}(\kappa_x,-\kappa_y)\right] \tag{3.30}$$

where j = f,g.

For homogeneity of the real-valued stochastic field given by Eq. 3.30, it is easily show that the following requirements exist:

$$E\left[dU_j^{(n)}(\kappa_x,\kappa_y)dU_k^{(m)}(\kappa_x',\kappa_y')\right] = 0 \tag{3.31}$$

If $n \neq m$ or if $n = m$ $(n,m = 1,2)$ and the regions (κ_x,κ_y) and (κ_x',κ_y') are disjointed, and

$$E\left[dU_j^{(1)}(\kappa_x,\kappa_y)dU_j^{(2)}(\kappa_x,\kappa_y)\right] = 0 \tag{3.32}$$

and also

$$E\left[dU_j^{(1)}(\kappa_x,\kappa_y)dU_k^{(1)}(\kappa_x,\kappa_y)\right] = E\left[dU_j^{(2)}(\kappa_x,\kappa_y)dU_k^{(2)}(\kappa_x,\kappa_y)\right] = = dF_{jk}^{(1)}(\kappa_x,\kappa_y) \tag{3.33}$$

$$E\left[dU_j^{(2)}(\kappa_x,\kappa_y)dU_k^{(1)}(\kappa_x,\kappa_y)\right] = - E\left[dU_j^{(1)}(\kappa_x,\kappa_y)dU_k^{(2)}(\kappa_x,\kappa_y)\right] = dF_{jk}^{(2)}(\kappa_x,\kappa_y) \tag{3.34}$$

The second equalities in Eqs. 3.33 and 3.34, respectively, are derived from Eqs. 3.19 and 3.21. It is not hard to show Eq. 3.29, using the spectral representation of the field j(x,y) given by Eq. 3.30 and the orthogonal conditions for the real-valued increments $dU_j^{(1)}$ and $dU_j^{(2)}$ given by Eqs. 3.31-3.34. In fact,

$$R_{jk}^{(1)}(\xi_x,\xi_y) = E[j^{(1)}(x+\xi_x,y+\xi_y)k^{(1)}(x,y)]$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty E[\cos\{\kappa_x(x+\xi_x)+\kappa_y(y+\xi_y)\}dU_j^{(1)}(\kappa_x,\kappa_y)$$

$$+ \sin\{\kappa_x(x+\xi_x)+\kappa_y(y+\xi_y)dU_j^{(2)}(\kappa_x,\kappa_y)$$

$$+ \cos\{\kappa_x(x+\xi_x)-\kappa_y(y+\xi_y)\}dU_j^{(1)}(\kappa_x,-\kappa_y)$$

$$+ \sin\{\kappa_x(x+\xi_x) - \kappa_y(y+\xi_y)\}dU_j^{(2)}(\kappa_x,-\kappa_y)]$$

$$\times [\cos\{\kappa_x'(x+\xi_x)+\kappa_y'(y+\xi_y)\}dU_j^{(1)}(\kappa_x',\kappa_y')$$

$$+ \sin\{\kappa_x'(x+\xi_x)+\kappa_y'(y+\xi_y)\}dU_j^{(2)}(\kappa_x',\kappa_y')$$

$$+ \cos\{\kappa_x'(x+\xi_x)-\kappa_y'(y+\xi_y)\}dU_j^{(1)}(\kappa_x',-\kappa_y')$$

$$+ \sin\{\kappa_x'(x+\xi_x) - \kappa_y'(y+\xi_y)\}dU_j^{(2)}(\kappa_x',-\kappa_y')]$$

$$= \int_0^\infty\int_0^\infty \cos\{\kappa_x(x+\xi_x)+\kappa_y(y+\xi_y)\}\cos(\kappa_x x+\kappa_y y)E[dU_j^{(1)}(\kappa_x,\kappa_y)dU_k^{(1)}(\kappa_x,\kappa_y)]$$

$$+ \cos\{\kappa_x(x+\xi_x)+\kappa_y(y+\xi_y)\}\sin(\kappa_x x+\kappa_y y)E[dU_j^{(1)}(\kappa_x,\kappa_y)dU_k^{(2)}(\kappa_x,\kappa_y)]$$

$$+ \sin\{\kappa_x(x+\xi_x)+\kappa_y(y+\xi_y)\}\cos(\kappa_x x+\kappa_y y)E[dU_j^{(2)}(\kappa_x,\kappa_y)dU_k^{(1)}(\kappa_x,\kappa_y)]$$

$$+ \sin\{\kappa_x(x+\xi_x)+\kappa_y(y+\xi_y)\sin(\kappa_x x+\kappa_y y)E[dU_j^{(2)}(\kappa_x,\kappa_y)dU_k^{(2)}(\kappa_x,\kappa_y)]]$$

+ with the $(\kappa_x,-\kappa_y)$ terms in the same form as in the above elements

$$= \int_0^\infty\int_0^\infty [\cos(\kappa_x\xi_x+\kappa_y\xi_y)dF_{jk}^{(1)}(\kappa_x,\kappa_y)+\sin(\kappa_x\xi_x+\kappa_y\xi_y)dF_{jk}^{(2)}(\kappa_x,\kappa_y)]$$

$$+ \int_0^\infty\int_0^\infty [\cos(\kappa_x\xi_x-\kappa_y\xi_y)dF_{jk}^{(1)}(\kappa_x,-\kappa_y)+\sin(\kappa_x\xi_x-\kappa_y\xi_y)dF_{jk}^{(2)}(\kappa_x,-\kappa_y)] \quad (3.35)$$

4. SIMULATION METHOD

Consider the simulation problem of the homogeneous stochastic fields $f(x,y)$ and $g(x,y)$ under the condition that the power spectral density function $S_{jk}(\kappa_x,\kappa_y)$ is specified such that

$$\begin{bmatrix} S_{ff}(\kappa_x,\kappa_y) & S_{fg}(\kappa_x,\kappa_y) \\ S_{gf}(\kappa_x,\kappa_y) & S_{gg}(\kappa_x,\kappa_y) \end{bmatrix}$$

$$S_{gf}(\kappa_x,\kappa_y) = \bar{S}_{fg}(\kappa_x,\kappa_y) \tag{4.1}$$

Since the power spectral density function constitutes a Hermitian and non-negative definite matrix, Eq. 4.1 can be decomposed as

$$\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \bar{a}_{11} & \bar{a}_{21} \\ 0 & \bar{a}_{22} \end{bmatrix} = \begin{bmatrix} |a_{11}|^2 & a_{11}\bar{a}_{21} \\ a_{21}\bar{a}_{11} & |a_{21}|^2+|a_{22}|^2 \end{bmatrix} \tag{4.2}$$

where each element a_{jk} can be obtained as

$$a_{11} = \sqrt{S_{ff}(\kappa_x,\kappa_y)}\; e^{i\phi_2(\kappa_x,\kappa_y)} = |a_{11}|^2 e^{i\phi_2(\kappa_x,\kappa_y)}$$

$$a_{21} = \frac{|S_{fg}(\kappa_x,\kappa_y)|}{\sqrt{S_{ff}(\kappa_x,\kappa_y)}}\; e^{i\{\phi_1(\kappa_x,\kappa_y)+\alpha_{21}(\kappa_x,\kappa_y)\}} = |a_{21}|^2\, e^{i\{\phi_1(\kappa_x,\kappa_y)+\alpha_{21}(\kappa_x,\kappa_y)\}}$$

$$a_{22} = \sqrt{\frac{S_{ff}(\kappa_x,\kappa_y)S_{gg}(\kappa_x,\kappa_y) - |S_{fg}(\kappa_x,\kappa_y)|^2}{S_{ff}(\kappa_x,\kappa_y)}}\; e^{i\phi_2(\kappa_x,\kappa_y)} = |a_{22}|\; e^{i\phi_2(\kappa_x,\kappa_y)} \tag{4.3}$$

where $\phi_1(\kappa_x,\kappa_y)$ and $\phi_2(\kappa_x,\kappa_y)$ are arbitrary phase angles and

$$\alpha_{21}(\kappa_x,\kappa_y) = \tan^{-1}\left\{\frac{\mathrm{Im}[S_{fg}(\kappa_x,\kappa_y)]}{\mathrm{Re}[S_{fg}(\kappa_x,\kappa_y)]}\right\} \tag{4.4}$$

On the other hand, the covariance of the orthogonal increment $dZ_j(\kappa_x,\kappa_y)$ is given by Eq. 1.19b, that is,

$$E[dZ_f \ \overline{dZ}_f] = S_{ff}(\kappa_x,\kappa_y)d\kappa_x d\kappa_y$$
$$E[dZ_f \ \overline{dZ}_g] = S_{fg}(\kappa_x,\kappa_y)d\kappa_x d\kappa_y \qquad (4.5)$$
$$E[dZ_g \ \overline{dZ}_g] = S_{gg}(\kappa_x,\kappa_y)d\kappa_x d\kappa_y$$

Comparison of Eqs. 4.2 and 4.5 motivates the introduction of a new definition for the orthogonal increments dZ_f and dZ_g to efficiently express the orthogonal increments in terms of the power spectral density functions, such that

$$dZ_f = dZ_{ff} \qquad (4.6)$$

$$dZ_g = dZ_{gf} + dZ_{gg} \qquad (4.7)$$

where

$$E[dZ_{ff} \ \overline{dZ}_{gg}] = E[dZ_{gf} \ \overline{dZ}_{gg}] = 0 \qquad (4.8)$$

And, similar to Eq. 1.22,

$$dZ_f = \frac{1}{2}\left[dU_{ff}^{(1)} - idU_{ff}^{(2)}\right] \qquad (4.9)$$

$$dZ_g = \frac{1}{2}\left[dU_{gf}^{(1)} - idU_{gf}^{(2)}\right] + \frac{1}{2}\left[dU_{gg}^{(1)} - idU_{gg}^{(2)}\right] \qquad (4.10)$$

where $dU_{jk}^{(1)}$ and $dU_{jk}^{(2)}$ are real-valued. Substitution of Eqs. 4.6 and 4.7 into Eq. 4.11, taking into account Eq. 8, yields

$$\begin{bmatrix} E[dZ_f \ \overline{dZ}_f] & E[dZ_f \ \overline{dZ}_g] \\ E[dZ_g \ \overline{dZ}_f] & E[dZ_g \ \overline{dZ}_g] \end{bmatrix} = \begin{bmatrix} E[|dZ_{ff}|^2] & E[dZ_{ff} \ \overline{dZ}_{gf}] \\ E[dZ_{gf} \ \overline{dZ}_{ff}] & E[|dZ_{gf}|^2 + |dZ_{gg}|^2] \end{bmatrix} \qquad (4.11)$$

By comparison of Eqs. 4.2 and 4.9, the orthogonal increments are expressed in terms of a_{ij} which in turn is a function of the power spectral density function such that

$$dZ_{ff}(\kappa_x,\kappa_y) = |a_{11}|\sqrt{d\kappa_x d\kappa_y}\ e^{i\phi_1(\kappa_x,\kappa_y)}$$

$$dZ_{gf}(\kappa_x,\kappa_y) = |a_{21}|\sqrt{d\kappa_x d\kappa_y}\; e^{i\{\alpha w_2(\kappa_x,\kappa_y)+\alpha_{21}(\kappa_x,\kappa_y)\}}$$

$$dZ_{gg}(\kappa_x,\kappa_y) = |a_{22}|\sqrt{d\kappa_x d\kappa_y}\; e^{i\phi_2(\kappa_x,\kappa_y)} \tag{4.12}$$

Also, equation Eqs. 4.9 and 4.10 with Eq. 4.12,

$$dU_{ff}^{(1)}(\kappa_x,\kappa_y) = 2|a_{11}|\sqrt{d\kappa_x d\kappa_y}\,\cos\phi_1(\kappa_x,\kappa_y) \tag{4.13a}$$

$$dU_{ff}^{(2)}(\kappa_x,\kappa_y) = -\,2|a_{11}|\sqrt{d\kappa_x d\kappa_y}\,\sin\phi_1(\kappa_x,\kappa_y) \tag{4.13b}$$

$$dU_{gf}^{(1)}(\kappa_x,\kappa_y) = 2|a_{21}|\sqrt{d\kappa_x d\kappa_y}\,\cos\{\phi_2(\kappa_x,\kappa_y) + \alpha_{21}(\kappa_x,\kappa_y)\} \tag{4.13c}$$

$$dU_{gf}^{(2)}(\kappa_x,\kappa_y) = -\,2|a_{21}|\sqrt{d\kappa_x d\kappa_y}\,\sin\{\phi_1(\kappa_x,\kappa_y) + \alpha_{21}(\kappa_x,\kappa_y)\} \tag{4.13d}$$

$$dU_{gg}^{(1)}(\kappa_x,\kappa_y) = 2|a_{22}|\sqrt{d\kappa_x d\kappa_y}\,\cos\phi_2(\kappa_x,\kappa_y) \tag{4.13e}$$

$$dU_{gg}^{(2)}(\kappa_x,\kappa_y) = -\,2|a_{22}|\sqrt{d\kappa_x d\kappa_y}\,\sin\phi_2(\kappa_x,\kappa_y) \tag{4.13f}$$

In Eq. 4.12, the arbitrary phase angles $\phi_1(\kappa_x,\kappa_y)$ and $\phi_2(\kappa_x,\kappa_y)$ must be appropriate random functions so that the rando amplitude dZ_{jk} given by Eq.4.12 satisfy the orthogonal conditions given by Eqs. 1.18 and 1.19. If we choose the independent random phases uniformly distributed between 0 and 2π as $\phi_1(\kappa_w,\kappa_y)$ and $\phi_2(\kappa_x,\kappa_y)$, it is easy to show that Eq. 4.12 satisfies Eqs. 1.18 and 1.19. Hence, from Eqs. 1.10 and 1.11, the stochastic fields f(x,y) and g(x,y) are expressed as

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\kappa_x x+\kappa_y y+\phi_1(\kappa_x,\kappa_y))}\,|a_{11}(\kappa_x,\kappa_y)|\sqrt{dk_x d\kappa_y} \tag{4.14a}$$

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\kappa_x x+\kappa_y y+\phi_1(\kappa_x,\kappa_y)+\alpha_{21}(\kappa_x,\kappa_y))}\,|a_{21}(\kappa_x,\kappa_y)|\sqrt{d\kappa_x d\kappa_y}$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\kappa_x x+\kappa_y y+\phi_2(\kappa_2,\kappa_y))}\,|a_{22}(\kappa_x,\kappa_y)|\sqrt{d\kappa_2 d\kappa_y} \tag{4.14b}$$

standing for the limit

$$f(x,y) = \lim_{\substack{\Delta\kappa_x\to 0\\ \Delta\kappa_y\to 0}} \sum_{n=-N}^{N} \sum_{m=-M}^{M} e^{i\left(\kappa_{x_n} x+\kappa_{y_m} y+\phi_{1nm}\right)} \left|a_{11}(\kappa_{x_n} \kappa_{y_m})\right| \sqrt{\Delta\kappa_x \Delta\kappa_y} \quad (4.15a)$$

$$g(x,y) = \lim_{\substack{\Delta\kappa_x\to 0\\ \Delta\kappa_y\to 0}} \sum_{n=-N}^{N} \sum_{m=-M}^{M} \Big[e^{i\left(\kappa_{x_n} x+\kappa_{y_m} y+\phi_{1nm}+\alpha_{21}(\kappa_{x_n},\kappa_{y_m})\right)} \left|a_{21}(\kappa_{x_n},\kappa_{y_m})\right| \sqrt{\Delta\kappa_x \Delta\kappa_y}$$

$$+ e^{i\left(\kappa_{x_m} x+\kappa_{y_m} y+\phi_{2nm}\right)} \left|a_{22}(\kappa_{x_n},\kappa_{y_m})\right| \sqrt{\Delta\kappa_x \Delta\kappa_y}\Big] \quad (4.15b)$$

For real-valued fields, from Eqs. 1.30 and Eqs. 4.9, 4.10 and 4.13:

$$f^{(1)}(x,y) = \int_0^\infty\int_0^\infty 2\left|a_{11}(\kappa_x,\kappa_y)\right| \sqrt{d\kappa_x d\kappa_y}\, \cos\{\kappa_x x+\kappa_y+\phi_1(\kappa_x,\kappa_y)\}$$

$$+ \int_0^\infty\int_0^\infty 2\left|a_{11}(\kappa_x,-\kappa_y)\right| \sqrt{d\kappa_x d\kappa_y}\, \cos\{\kappa_x x-\kappa_y y+\phi_2(\kappa_x,-\kappa_y)\} \quad (4.16a)$$

and

$$g^{(1)}(x,y) = \int_0^\infty\int_0^\infty 2\left|a_{21}(\kappa_x,\kappa_y)\right| \sqrt{d\kappa_x d\kappa_y}\, \cos\{\kappa_x x+\kappa_y y+\phi_1(\kappa_x,\kappa_y)+\alpha_{21}(\kappa_x,\kappa_y)\}$$

$$+ \int_0^\infty\int_0^\infty 2\left|a_{21}(\kappa_x,-\kappa_y)\right| \sqrt{d\kappa_x d\kappa_y}\, \cos\{\kappa_x x-\kappa_y y+\phi_1(\kappa_x,-\kappa_y)+\alpha_{21}(\kappa_x,-\kappa_y)\}$$

$$+ \int_0^\infty\int_0^\infty 2\left|a_{22}(\kappa_x,\kappa_y)\right| \sqrt{d\kappa_x d\kappa_y}\, \cos\{\kappa_x x+\kappa_y y+\phi_2(\kappa_x,\kappa_y)\}$$

$$+ \int_0^\infty\int_0^\infty 2\left|a_{22}(\kappa_x,-\kappa_y)\right| \sqrt{d\kappa_x d\kappa_y}\, \cos\{\kappa_x x-\kappa_y y+\phi_2(\kappa_x,-\kappa_y)\} \quad (4.16b)$$

The integrals mean

$$f^{(1)}(x,y) = \lim_{\substack{\Delta\kappa_x \to 0 \\ \Delta\kappa_y \to 0}} \sum_{n=1}^{N} \sum_{m=1}^{M} 2|a_{11}(\kappa_{x_n} \kappa_{y_m})| \sqrt{\Delta\kappa_x \Delta\kappa_y} \cos\{\kappa_{x_n} x + \kappa_{y_m} y + \phi_{1_{nm}}\}$$

$$+ 2|a_{11}(\kappa_{x_n}, -\kappa_{y_m})| \sqrt{\Delta\kappa_x \Delta\kappa_y} \cos\{\kappa_{x_n} x - \kappa_{y_m} y + \phi'_{1_{nm}}\} \tag{4.17a}$$

and

$$g^{(1)}(x,y) = \lim_{\substack{\Delta\kappa_x \to 0 \\ \Delta\kappa_y \to 0}} \sum_{n=1}^{N} \sum_{m=1}^{M} 2|a_{21}(\kappa_{x_n}, \kappa_{y_m})| \sqrt{\Delta\kappa_x \Delta\kappa_y} \cos\{\kappa_{x_m} x + \kappa_{y_m} y + \phi_{1_{nm}} + \alpha_{21}(\kappa_{x_n}, \kappa_{y_m})\}$$

$$+ 2|a_{21}(\kappa_{x_n}, -\kappa_{y_m})| \sqrt{\Delta\kappa_x \Delta\kappa_y} \cos\{\kappa_{x_n} x - \kappa_{y_m} y + \phi'_{1nm} + \alpha_{21}(\kappa_{x_n}, -\kappa_{y_m})\}$$

$$+ 2|a_{22}(\kappa_{x_n}, \kappa_{y_m})| \sqrt{\Delta\kappa_x \Delta\kappa_y} \cos\{\kappa_{x_n} x + \kappa_{y_m} y + \phi_{2_{nm}}\}$$

$$+ 2|a_{22}(\kappa_{x_n}, \kappa_{y_m})| \sqrt{\Delta\kappa_x \Delta\kappa_y} \cos\{\kappa_{x_n} x + \kappa_{y_m} y + \phi_{2_{nm}}\}$$

$$+ 2|a_{22}(\kappa_{x_n} - \kappa_{y_m})| \sqrt{\Delta\kappa_x \Delta\kappa_y} \cos\{\kappa_{x_n} x - \kappa_{y_m} y + \phi'_{2_{nm}}\} \tag{4.17b}$$

If the fields $f^{(1)}(x,y)$ and $g^{(1)}(x,y)$ are quadrant symmetry where the power spectral density function satisfies $S_{jk}(\kappa_x, \kappa_y) = S_{jk}(\kappa_x, -\kappa_y)$, then

$$a_{jk}(\kappa_{x_n}, \kappa_{y_m}) = a_{jk}(\kappa_{x_n}, -\kappa_{y_m}) \qquad j,k=1,2 \tag{4.18a}$$

$$\alpha_{21}(\kappa_{x_n}, \kappa_{y_m}) = \alpha_{21}(\kappa_{x_n}, -\kappa_{y_m}) \tag{4.18b}$$

Hence, Eq. 4.17 reduces to, for quadrant symmetry,

$$f^{(1)}(x,y) = \lim_{\substack{\Delta\kappa_x\to 0\\ \Delta\kappa_y\to 0}} \sum_{n=1}^{N}\sum_{m=1}^{M} 2|a_{11}(\kappa_{x_n},\kappa_{y_m})|\sqrt{\Delta\kappa_x\Delta\kappa_y}\,[\cos\{\kappa_{x_n}x+\kappa_{y_m}y+\phi_{1_{nm}}\}$$

$$+\cos\{\kappa_{x_n}x-\kappa_{y_m}y+\phi'_{1_{nm}}\}] \tag{4.19a}$$

and

$$g^{(2)}(x,y) = \lim_{\substack{\Delta\kappa_x\to 0\\ \Delta\kappa_y\to 0}} \sum_{n=1}^{N}\sum_{m=1}^{M} 2|a_{21}(\kappa_{x_n},\kappa_{y_m})|\sqrt{\Delta\kappa_x\Delta\kappa_y}$$

$$\times\,[\cos\{\kappa_{x_n}x+\kappa_{y_m}y+\alpha_{21}(\kappa_{x_n},\kappa_{y_m})+\phi_{1_{nm}}\} \tag{4.19b}$$

$$+\cos\{\kappa_{x_n}x-\kappa_{y_m}y+\alpha_{21}(\kappa_{x_n},\kappa_{y_m})+\phi'_{1_{nm}}\}]$$

$$+\,2|a_{22}(\kappa_{x_n},\kappa_{y_m})|\sqrt{\Delta\kappa_x\Delta\kappa_y}\,[\cos\{\kappa_{x_n}x+\kappa_{y_m}y+\phi_{2_{nm}}\}+\cos\{\kappa_{x_n}x-\kappa_{y_m}y+\phi'_{2_{nm}}\}]$$

The above equations (Eqs. 4.17 and 4.19) are suitable for the computer simulation of $f^{91)}(x,y)$ and $g^{(1)}(x,y)$. Also, making use of the FFT (Fast Fourier Transform) technique to the summations appearing in Eqs. 4.17 and 4.19, drastically reduces the computing time.

5. TIME-SPACE STOCHASTIC FIELDS

In the previous sections we were concerned with stochastic fields whose sample functions are continuous functions of the space coordinates x and y. We now turn to a more general case where the stochastic fields are functions of time t as well as of the space coordinates x and y.

As defined in Sections 2 and 3, the complex-values time-space stochastic fields f(x,y,t) and g(x,y,t) can be defined as

$$f(x,y,t) = f^{(1)}(x,y,t) + if^{(2)}(x,y,t) \tag{5.1}$$

$$g(x,y,t) = g^{(1)}(x,y,t) + ig^{(2)}(x,y,t) \tag{5.2}$$

where $f^{(1)}$, $f^{(2)}$, $g^{(1)}$ and $g^{(2)}$ are real-valued stochastic fields. The mean is defined such that

$$E[f(x,y,t)] = E[f^{(1)}(x,y,t)] + iE[f^{(2)}(x,y,t)] \tag{5.3}$$

$$E[g(x,y,t)] = E[g^{(1)}(x,y,t)] + iE[g^{(2)}(x,y,t)] \tag{5.4}$$

Now suppose that the fields are stationary, homogeneous and zero-mean. Then, the time-space covariance function of the fields can be written in matrix form as

$$\underline{Q}(\xi_x,\xi_y,\tau) = \begin{bmatrix} Q_{ff}(\xi_x,\xi_y,\tau), Q_{fg}(\xi_x,\xi_y,\tau) \\ \\ Q_{gf}(\xi_x,\xi_y,\tau), Q_{gg}(\xi_x,\xi_y,\tau) \end{bmatrix} = \tag{5.5}$$

$$= \begin{bmatrix} E[f(x+\xi_x,y+\xi_y,t+\tau)\overline{f}(x,y,t)], E[f(x+\xi_x,y+\xi_y,t+\tau)\overline{g}(x,y,t)] \\ \\ E[g(x+\xi_x,y+\xi_y,t+\tau)\overline{f}(x,y,t)], E[g(x+\xi_x,y+\xi_y,t+\tau)\overline{g}(x,y,t)] \end{bmatrix}$$

From the definition given by Eq. 3.5, the time-space covariance function possesses the property such that

$$Q_{jk}(\xi_x,\xi_y,\tau) = \overline{Q}_{kj}(-\xi_x,-\xi_y,-\tau) \qquad j,k = f,g \tag{5.6}$$

that is, the covariance matrix is Hermitian. Transforming the time lag τ into the frequency ω by means of the Wiener-Khintchine transform yields the temporal frequency spatial spectral density function $P_{jk}(\xi_x,\xi_y,\omega)$ as a function of ξ:

$$P_{jk}(\xi_x,\xi_y,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau}\, Q_{jk}(\xi_x,\xi_y,\tau)d\tau \tag{5.7}$$

By performing an inverse transformation, one can reclaim $Q_{jk}(\xi_x,\xi_y,\tau)$ as

$$Q_{jk}(\xi_x,\xi_y,\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau}\, P_{jk}(\xi_x,\xi_y,\omega)d\omega \tag{5.8}$$

If the separation distances ξ_x and ξ_y are zero, the temporal frequency spatial cross-spectral density function (at any point x and y):

$$P_{jk}(0,0,\omega) \equiv S_{jk}(\omega) \tag{5.9}$$

Normalization of the temporal frequency cross-spectral density function with respect to its value at $\xi_x = \xi_y = 0$ gives the frequency-dependent spatial covariance function as follows:

$$\gamma_{jk}(\xi_x,\xi_y,\omega) = \frac{P_{jk}(\xi_x,\xi_y,\omega)}{S_{jk}(\omega)} \tag{5.10}$$

The spatial covariance function $R_{jk}(\xi_x,\xi_y)$ can be defined as

$$R_{jk}(\xi_x,\xi_y) = Q_{jk}(\xi_x,\xi_y,0) \tag{5.11}$$

and hence, with the aid of Eqs. 5.8 and 5.10,

$$R_{jk}(\xi_x,\xi_y) = \int_{-\infty}^{\infty} P_{jk}(\xi_x,\xi_y,\omega)d\omega = \int_{-\infty}^{\infty} S_{jk}(\omega)\gamma_{jk}(\xi_x,\xi_y,\omega)d\omega \tag{5.12}$$

Thus, the spatial covariance function is a weighted integral of the frequency-dependent spatial covariance function $\gamma_{jk}(\xi_x,\xi_y,\omega)$ with the point cross-spectral density function $S_{jk}(\omega)$ as the weight. If the time-space stochastic fields are assumed to be ergodic, the spatial covariance function may be estimated from the temporal average such that

$$R_{jk}(\xi_x,\xi_y) = Q_{jk}(\xi_x,\xi_y,0) = \lim_{T\to 0} \int_0^T j^{(n)}(x+\xi_x,y+\xi_y,t)\overline{k}^{(n)}(x,y,t)dt \tag{5.13a}$$

or from the following spatial average

$$R_{jk}(\xi_x,\xi_y) = Q_{jk}(\xi_x,\xi_y,0) = \lim_{\substack{L_x\to\infty \\ L_y\to\infty}} \frac{1}{L_xL_y} \int_0^{L_x}\int_0^{L_y} j^{(n)}(x+\xi_x,y+\xi_y t)\overline{k}^{(n)}(x,y,t)dxdy \tag{5.13b}$$

in which $j^{(n)}(x,y,t)$ represents the n-th sample function of $j(x,y,t)$, where $j = f, g$. However, in practice, Eq. 5.13b cannot usually be used since the observations $f^{(n)}(x,y,t)$ are made at only a few discrete locations along the x and y axes and therefore the integration of Eq. 5.13b is not possible.

Similarly, transforming the separation distances ξ_x and ξ_y into the wave numbers κ_x and κ_y by means of the Wiener-Khintchine transform gives the temporal covariance spatial cross-wave number spectral density function $V_{jk}(\kappa_x,\kappa_y,\tau)$ as a function of τ:

$$V_{jk}(\kappa_x,\kappa_y,\tau) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\kappa_x\xi_x+\kappa_y\xi_y)} Q_{jk}(\xi_x,\xi_y,\tau)d\xi_x d\xi_y \tag{5.14}$$

By performing an inverse transformation, one can reclaim $Q_{jk}(\xi_x,\xi_y,\tau)$ as

$$Q_{jk}(\xi_x,\xi_y,\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\kappa_x\xi_x+\kappa_y\xi_y)} V_{jk}(\kappa_x,\kappa_y,\tau)d\kappa_x d\kappa_y \tag{5.15}$$

Finally, transforming both the time lag τ and the separation distances ξ_x and ξ_y into the frequency ω and wave numbers κ_x and κ_y by means of the Wiener-Khintchine transform gives the frequency number cross-spectral density function $S_{jk}(\kappa_x,\kappa_y,\omega)$ such that

$$S_{jk}(\kappa_x,\kappa_y,\omega) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\kappa_x\xi_x+\kappa_y\xi_y+\omega\tau)} Q_{jk}(\xi_x,\xi_y,\tau)d\xi_x d\xi_y d\tau \quad (5.16)$$

from the inverse transformation

$$Q_{jk}(\xi_x,\xi_y,\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\kappa_x\xi_x+\kappa_y\xi_y+\omega\tau)} S_{jk}(\kappa_x,\kappa_y,\omega)d\kappa_x d\kappa_y d\omega \quad (5.17)$$

Due to Eqs. 5.7, 5.8 and 5.14, the frequency wave number spectral density function $S_{jk}(\kappa_x,\kappa_y,\omega)$ is also related to $P_{jk}(\xi_x,\xi_y,\omega)$ and $V_{jk}(\kappa_x,\kappa_y,\tau)$ such that

$$S_{jk}(\kappa_x,\kappa_y,\omega) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\kappa_x\xi_x+\kappa_y\xi_y)} P_{jk}(\xi_x,-\xi_y,\omega)d\xi_x d\xi_y \quad (5.18)$$

and

$$S_{jk}(\kappa_x,\kappa_y,\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega\tau} V_{jk}(\kappa_x,\kappa_y,\tau)d\tau \quad (5.19)$$

By the inverse transformation

$$P_{jk}(\kappa_x,\kappa_y,\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\kappa_x\xi_x+\kappa_y\xi_y)} S_{jk}(\kappa_x,\kappa_y,\omega)d\kappa_x d\kappa_y \quad (5.20)$$

$$V_{jk}(\kappa_x,\kappa_y,\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} S_{jk}(\kappa_x,\kappa_y,\omega)d\omega \quad (5.21)$$

And the point spectral density function $S_{jk}(\omega)$ defined by Eq. 5.9 is also written as

$$S_{jk}(\omega) \equiv P_{jk}(0,0,\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{jk}(\kappa_x,\kappa_y,\omega)d\kappa_x d\kappa_y \tag{5.22}$$

Similar to Eq. 5.10, normalization of the frequency wave number spectral density function $S_{jk}(\kappa_x,\kappa_y,\omega)$ with respect to the point spectral density function $S_{jk}(\omega)$ yields the frequency-dependent wave number spectral density function as follows:

$$\tilde{\gamma}_{jk}(\kappa_x,\kappa_y,\omega) = \frac{S_{jk}(\kappa_x,\kappa_y,\omega)}{S_{jk}(\omega)} \tag{5.23}$$

The spatial wave number spectral density function $S_{jk}(\kappa_x,\kappa_y)$ can be defined as $S_{jk}(\kappa_x,\kappa_y) = V_{jk}(\kappa_x,\kappa_y,0)$ and hence, with the aid of Eqs. 5.21 and 5.23,

$$S_{jk}(\kappa_x,\kappa_y) = V_{jk}(\kappa_x,\kappa_y,0) =$$

$$= \int_{-\infty}^{\infty} S_{jk}(\kappa_x,\kappa_y,\omega)d\omega = \int_{-\infty}^{\infty} S_{jk}(\omega)\tilde{\gamma}_{jk}(\kappa_x,\kappa_y,\omega)d\omega \tag{5.25}$$

As described above, there is a close relationship among the various functions which are summarized in Fig. 1. From a general characterization point of view, the frequency wave number spectral density function $S_{jk}(\kappa_x,\kappa_y,\omega)$ defined by Eq. 5.16 may be more useful because this function plays a central role when we perform an analysis similar to that used for the spectral representation of stochastic fields as described in Sections 2 and 3. On the other hand, the spatial covariance function $R_{jk}(\xi_x,\xi_y)$ and the spatial spectral density function $S_{jk}(\kappa_x,\kappa_y)$ are also important functions to characterize the spatial variation of the time-space stochastic fields $j(x,y,t)$ and $k(x,y,t)$. In fact, the spatial variation of earthquake ground motion displacements is of major significance for the resopnse of underground lifeline structures such as pipelines. In Section 5, a numerical example for the spatial variation of earthquake ground displacement is presented.

6. NUMERICAL EXAMPLE*

To visually illustrate the significance of the simulation equations previously described, we present here several numerical examples of the sample function of $f(x,y)$ simulated using the simulation equations. And also we present here an example of the seismic ground deformation pattern $u(x,y,t_0)$ of a ground surface at time = t_0 estimated fvrom seismic array observation in Taiwan (SMART-1 Array).

6.1 Simulation Examples

For simplicity, consider the simulation of $f^{(1)}(x,y)$ using Eq. 2.19a (quadrant symmetry, uni-variate, two-dimensional case). From Eq. 2.19a:

$$f^{(1)}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} 2\sqrt{S_{ff}(\kappa_{x_n},\kappa_{y_m})\Delta\kappa_x\Delta\kappa_y}\ [\cos(\kappa_{x_n}x+\kappa_{y_m}y+\phi_{1_{nm}}) + \cos(\kappa_{x_n}x-\kappa_{y_m}y+\phi'_{1_{nm}})] \tag{6.1a}$$

$$\Delta\kappa_x = \frac{\kappa_{xu}}{N}, \qquad \Delta\kappa_y = \frac{\kappa_{yu}}{M}, \qquad \kappa_{x_n} = n\Delta\kappa_x, \qquad \kappa_{y_m} = m\Delta\kappa_y \tag{6.1b}$$

* Note: The figures and computer codes in this section were provided by F. Yamazaki (Visiting Scholar, Columbia University), Research Engineer, Ohsaki Research Institute, Shimizu Construction Co., Ltd., Tokyo, Japan.

where $\phi_{1_{nm}}$ and $\phi'_{1_{nm}}$ are mutually independent random phase angles, uniformly distributed between 0 and 2π.

Equation 6.1 signifies that a sample function $f^{(1)}(x,y)$ can be expressed as the sum of many elementary waves $\cos(\kappa_{x_n} x + \kappa_{y_m} y + \phi_{1_{nm}})$ and $\cos(\kappa_{x_n} x - \kappa_{y_m} y + \phi'_{1_{nm}})$ which propagate in the A and B directions, respectively, as shown in Fig. 1 with amplitude $2\sqrt{S_{ff}(\kappa_{x_n}, \kappa_{y_m})\Delta\kappa_x \Delta\kappa_y}$. To illustrate the above, two sample functions are generated using only the first term of Eq. 6.1a.

In Fig. 2a, a sample function of $f^{(1)}(x,y)$ is plotted for an isotropic power spectral density function (see Fig. 3) such as

$$S_{ff}(\kappa) = \sigma^2 \frac{b^2}{4\pi} e^{-\left(\frac{b\kappa}{e}\right)^2} \tag{6.2a}$$

$$\kappa = \sqrt{\kappa_x^2 + \kappa_y^2} \tag{6.2b}$$

For the numerical example, the following data are used: $\sigma = 1$, $b = 1$, $M = N = 64$, $\kappa_u = 2\pi$. From Fig. 2a, a sample function exhibits an isotropic pattern where the variation pattern is independent of direction. However, if we use only the first term in Eq. 6.1a for the simulation of $f^{(1)}(x,y)$, a directional-dependent pattern as shown in Fig. 2b is observed, notwithstanding the use of an isotropic power spectral density function.

6.2 Ground Deformation Using Seismic Array Observations

The data used in this study consist of the original accelerograms recorded on January 29, 1981 (Event 5) by a SMART-1 seismograph array (see Fig. 4) installed at Lotung, Taiwan. In this study, a displacement time history along the direction ($\phi = 77°$ or N13°W) which is considered to be approximately the direction of the seismic source of this earthquake (Event 5), is computed at each accelerogram station from two-component data (EW and NS). The purpose of this study is to estimate the spatial variation of seismic ground displacement for the analysis of underground pipelines. A more detailed description of this study is given in a report by T. Harada and T. Oda (1984).

By interpreting the displacement time history u(x,y,t) (N13°W component) at each station as sample functions of the uni-variate and spatially two-dimensional time-space stochastic process $f^{(1)}(x,y,t)$, and using Eq. 3.13a, a spatial correlation function $R_{uu}(\xi_x,\xi_y)$ was computed from the records of all the combinations of the 17 stations, C-00, I-03 -- I-12, M-03 -- M-09 and $\overline{0}$-04 -- $\overline{0}$-09, specifying the standard stations as C-00, M-05 and $\overline{0}$-05. Since the computed correlation functions approximately indicate quadrant symmetric behavior $\left(R_{uu}(\xi_x,\xi_y) = R_{uu}(\xi_x,-\xi_y)\right)$, all the correlation coefficient data were plotted by slim arrows as shown in Fig. 5. By judging the distribution of the correlation coefficients, a simple analytical correlation function was assumed such that

$$R_{uu}(\xi_x,\xi_y) = \sigma_{uu}^2\left[1 - 2\left(\frac{\xi_x}{b_x}\right)^2\right]\exp\left[-\left(\frac{\xi_x}{b_x}\right)^2 - \left(\frac{\xi_y}{b_y}\right)^2\right] \qquad \xi_x,\ \xi_y \geq 0 \qquad (6.3)$$

where σ_{uu} = 1.24 (cm), $b_x = 1.131 \times 10^3$ (m), $b_y = 3.012 \times 10^3$ (m). The values of Eq. 6.3 are also plotted with fat arrows in Fig. 5 indicating that the analytical form in Eq. 6.3 is approximately valid. From the Wiener-Khintchine relationship given by Eqs. 1.15 and 1.16, the corresponding power spectral density function $S_{uu}(\kappa_x,\kappa_y)$ is obtained as

$$S_{uu}(\kappa_x,\kappa_y) = \frac{\sigma_{uu}^2}{8\pi}\, b_x^3 b_y \kappa_x^2\, \exp\left[-\left(\frac{b_x\kappa_x}{2}\right)^2 - \left(\frac{b_y\kappa_y}{2}\right)^2\right] \qquad \kappa_x,\ \kappa_y \geq 0 \qquad (6.4)$$

A sample function of u(x,y) in the area of 22747.60 (m) x 19884.06 (m) is shown in Fig. 6. In this example, the following data are used in Eq. 6.1: M = N = 64, $\kappa_{xu} = 10/b_x = 8.84 \times 10^{-3}$(rad/m), $\kappa_{yn} = 10/b_y = 3.32 \times 10^{-3}$ (rad/m).

It is observed from Fig. 6 that there is relatively rapid variation along the x-axis (N13°W, seismic source approximate direction) compared with the variation along the y axis. From the number of peaks (8) along the x-axis (22747.6 m) in Fig. 6, the apparent wave length along the x-axis is estimated to be about 2.8 (km) (22747.618). Hence, the pattern in Fig. 6 indicates that a single wave with a wavelength of approximately 2.8 (km) propagates in the x-direction. In fact, for Event 5 data, the other study shows that a strong portion of the records consist of a wave with frequency of approximately 1 (Hz) and that it propagates in the x-direction with a speed of about 3 (km/s) ([B.A. Bolt, et al., 1982] indicating a wavelength of 3 (km) (3/1). This result is quite consistent with the variaiton pattern shown in Fig. 6.

7. ESTIMATION OF SPECTRAL DENSITY FUNCTION

7.1 Bi-Variate, One-Dimensional Case

In this section we considerthe estimation of a spectral density function from the finite-length real-valued records f(x) and g(x) with zero mean defined in the range $0 \leqq x \leqq L$.

The finite-range Fourier transform can be defined such that [Bendat and Piersol, 1971),

$$F_j(\kappa,L) = \int_0^L j(x)e^{-i\kappa x}\, dx \tag{7.1a}$$

$$F_j(-\kappa,L) = \overline{F}_j(\kappa,L) \qquad j = f,\ g \tag{7.1b}$$

Assuming that f(x) and g(x) are sampled at N equally spaced points with distance Δx apart, then f(x) and g(x) can be expressed as

$$j(n) = j(n\Delta x) \qquad n=1,2,\ldots,N \tag{7.2}$$

For arbitrary κ, the discrete version of Eq. 7.1 is

$$F_j(\kappa,L) = \Delta x \sum_{n=1}^{N} j(n)e^{-i\kappa n\Delta x} \tag{7.3}$$

The usual selection of a discrete wave number for the computation of $F_j(\kappa,L)$ is

$$\kappa_p = \frac{2\pi p}{L} = \Delta\kappa p = \frac{2\pi p}{N\Delta x} \qquad p=1,2,\ldots,N \tag{7.4}$$

At these wave numbers, Eq. 3 can be written as

$$F_j(\kappa_p,L) = \Delta x \sum_{n=1}^{N} j(n)e^{-i\frac{2\pi pn}{N}} \qquad p=1,2,\ldots,N \tag{7.5}$$

On the other hand, for large L, the covariance function $R_{jk}(\xi)$ may be estimated by

$$R_{jk}(\xi) = \frac{1}{L}\int_0^{L-3} j(x+\xi)k(x)dx \qquad 0 \leq \xi \leq L \tag{7.6a}$$

$$R_{jk}(\xi) = \frac{1}{L}\int_{-\xi}^{L} j(x+\xi)k(x)dx \qquad -L \leq \xi < 0 \tag{7.6b}$$

Recalling the Wiener-Khintchine relationships, the power spectral density function $S_{jk}(\kappa)$ may also be estimated by

$$S_{jk}(\kappa) = \frac{1}{2\pi}\int_{-L}^{L} R_{jk}(\xi)e^{-i\kappa\xi}\, d\xi \tag{7.7}$$

Equation 7.7 is also writen substituting Eq. 7.6 into Eq. 7.7 as

$$S_{jk}(\kappa) = \frac{1}{2\pi L} \Big[\int_{-L}^{0} \Big\{\int_{-\xi}^{L} j(x+\xi)k(x)dx\Big\} e^{-i\kappa\xi}\, d\xi + \int_{0}^{L} \Big\{\int_{0}^{L-\xi} j(x+\xi)k(x)dx\Big\} e^{-i\kappa\xi}\, d\xi \tag{7.8}$$

By changing the region of integration from (x,ξ) to (x,β) where $\beta = \xi+x$, $d\xi = d\beta$, the above integral can be expressed as

$$\int_{-L}^{0}\int_{-\xi}^{L} dxd\xi + \int_{0}^{L}\int_{0}^{L} dxd\xi = \int_{0}^{L}\int_{0}^{L} dxd\beta \tag{7.9}$$

Hence, Eq. 7.8 becomes the accounting from Eq. 7.9 such that

$$S_{jk}(\kappa) = \frac{1}{2\pi L} \int_{0}^{L} j(\beta)e^{-i\kappa\beta}\, d\beta \int_{0}^{L} k(x)e^{i\kappa x}\, dx \tag{7.10}$$

Recalling Eq. 7.1, Eq. 7.10 can also be expresed as

$$S_{jk}(\kappa) = \frac{1}{2\pi L} F_j(\kappa,L) \cdot \overline{F}_k(\kappa,L) \tag{7.11}$$

At the discrete wave number, Eq. 11 is expressed using Eqs. 7.4 and 7.5 as follows.

$$S_{jk}(\kappa_p) = \frac{1}{\Delta\kappa} \Big[\frac{1}{N} \sum_{n=1}^{N} j(n)e^{-i\kappa_p n\Delta x}\Big]\Big[\frac{1}{N} \sum_{n=1}^{N} k(n)e^{i\kappa_p n\Delta x}\Big] \tag{7.12}$$

If j = k, Eq. 7.12 reduces to

$$S_{jj}(\kappa_p) = \frac{1}{\Delta\kappa} \left|\frac{1}{N} \sum_{n=1}^{N} j(n)e^{-i\kappa_p n\Delta x}\right|^2 \tag{7.13}$$

Equations 7.12 and 7.13 are suitable for the estimation of the power spectral density function via finite Fourier transforms using the Fast Fourier Transform technique.

7.2 Bi-Variate, Two-Dimensional Case

Using procedures similar to that for the bi-variate, one-dimensional case described in the previous section, we describe the estimation of the power spectral density function for the bi-variate, two-dimensional, real-valued stochastic fields f(x,y) and g(x,y), defined in the finite regions $0 \leq x \leq L_x$ and $0 \leq y \leq L_y$.

The finite-range Fourier transform for the two-dimensional case can be defined such that

$$F_j(\kappa_x,\kappa_y,L_x,L_y) = \int_0^{L_x}\int_0^{L_y} j(x,y)e^{-i(\kappa_x x+\kappa_y y)}\,dxdy \qquad (7.14a)$$

$$F_j(-\kappa_x,-\kappa_y,L_x,L_y) = \overline{F}_j(\kappa_x,\kappa_y,L_x,L_y) \qquad j=f,g \qquad (7.14b)$$

and, at discrete wave numbers and points, Eq. 7.14 can be expressed as

$$F_j(\kappa_{x_p},\kappa_{y_q},L_x,L_y) = \Delta x\Delta y \sum_{n=1}^{N}\sum_{m=1}^{M} j(n,m)e^{-i\left(\frac{2\pi pn}{N}+\frac{2\pi qm}{M}\right)} \qquad (7.15)$$

where p = 1,2,...,N, q = 1,2,...,M and

$$\Delta x = \frac{L_x}{N}\ ,\ \Delta y = \frac{L_y}{M} \qquad (7.16a)$$

$$\kappa_{x_p} = \frac{2\pi p}{L_x} = \Delta\kappa_x p = \frac{2\pi p}{N\Delta x}\ , \qquad \kappa_{y_q} = \frac{2\pi q}{L_y} = \Delta\kappa_y q = \frac{2\pi q}{M\Delta y} \qquad (7.16b)$$

For large L_x, L_y, the covariance function $R_{jk}(\xi_x,\xi_y)$ may also be estimated by

$$R_{jk}(\xi_x,\xi_y) = \frac{1}{L_xL_y}\int_0^{L_x-\xi_x}\int_0^{L_y-\xi_y} j(x+\xi_x,y+\xi_y)k(x,y)dxdy \tag{7.17a}$$

for $0 \leq \xi_x \leq L_x$ and $0 \leq \xi_y \leq L_y$

$$R_{jk}(\xi_x,\xi_y) = \frac{1}{L_xL_y}\int_{-\xi_x}^{L_x}\int_0^{L_y-\xi_y} j(x+\xi_x,y+\xi_y)k(x,y)dxdy \tag{7.17b}$$

for $-L_x \leq \xi_x < 0$ and $0 \leq \xi_y \leq L_y$

$$R_{jk}(\xi_x,\xi_y) = \frac{1}{L_xL_y}\int_{-\xi_x}^{L_x}\int_{-\xi_y}^{L_y} j(x+\xi_x,y+\xi_y)k(x,y)dxdy \tag{7.17c}$$

for $-L_x \leq \xi_x \leq 0$ and $-L_y \leq \xi_y < 0$

$$R_{jk}(\xi_x,\xi_y) = \frac{1}{L_xL_y}\int_0^{L_x-\xi_x}\int_{-\xi_y}^{L_y} j(x+\xi_x,y+\xi_y)k(x,y)dxdy \tag{7.17d}$$

for $0 \leq \xi_x \leq L_x$ and $-L_y \leq \xi_y < 0$

Recalling the Wiener-Khintchine relationships, the power spectral density function $S_{jk}(\kappa_x,\kappa_y)$ may be estimated by

$$S_{jk}(\kappa_x,\kappa_y) = \frac{1}{(2\pi)^2}\int_{-L_x}^{L_x}\int_{L_y}^{L_y} R_{jk}(\xi_x,\xi_y)e^{-i(\kappa_x\xi_x+\kappa_y\xi_y)}\, d\xi_x d\xi_y \tag{7.18}$$

Substituting Eq. 7.17 into Eq. 7.18 and taking into account the following relationship similar to Eq. 7.9 with $\beta_x = \xi_x+x$ and $\beta_y = \xi_y + y$,

$$\int_0^{L_x}\int_0^{L_y}\int_0^{L_x}\int_0^{L_y} dxdyd\beta_x d\beta_y = \int_0^{L_x}\int_0^{L_x} dxd\beta_x \int_0^{L_y}\int_0^{L_y} dyd\beta_y$$

$$= \left\{\int_{-L_x}^{0}\int_{-\xi_x}^{L_x} dx d\xi_x + \int_{0}^{L_x}\int_{0}^{L_x-\xi_x} dx d\xi_x\right\} \times \left\{\int_{-L_y}^{0}\int_{-\xi_y}^{L_y} dy d\xi_y + \int_{0}^{L_y}\int_{0}^{L_y-\xi_y} d_y d\xi_y\right\}$$

$$= \int_{-L_x}^{0}\int_{-L_y}^{0}\int_{-\xi_x}^{L_x}\int_{-\xi_y}^{L_y} dx dy d\xi_x d\xi_y + \int_{-L_x}^{0}\int_{0}^{L_y}\int_{-\xi_x}^{L_x}\int_{0}^{L_y-\xi_y} dx dy d\xi_x d\xi_y$$

$$+ \int_{0}^{L_x}\int_{-L_y}^{0}\int_{0}^{L_x-\xi_x}\int_{-\xi_y}^{L_y} dx dy d\xi_x d\xi_y + \int_{0}^{L_x}\int_{0}^{L_y}\int_{0}^{L_x-\xi_x}\int_{0}^{L_y-\xi_y} dx dy d\xi_x d\xi_y \qquad (7.19)$$

Equation 7.18 is also expressed as

$$S_{jk}(\kappa_x,\kappa_y) = \frac{1}{(2\pi)^2 L_x L_y} \int_{0}^{L_x}\int_{0}^{L_y} j(\beta_x,\beta_y) e^{-i(\kappa_x\beta_x+\kappa_y\beta_y)} d\beta_x d\beta_y$$
$$\times \int_{0}^{L_x}\int_{0}^{L_y} k(x,y) e^{i(\kappa_x x+\kappa_y y)} dx dy \qquad (7.20)$$

Substitution of Eq. 7.14 into Eq. 7.20 yields

$$S_{jk}(\kappa_x,\kappa_y) = \frac{1}{(2\pi)^2 L_x L_y} F_j(\kappa_x,\kappa_y,L_x,L_y) \cdot \overline{F}_k(\kappa_x,\kappa_y,L_x,L_y) \qquad (7.21)$$

Using Eqs. 7.15 and 7.16, Eq. 7.21 is expressed in terms of the discrete form

$$S_{jk}{}^*(\kappa_{x_p},\kappa_{y_q}) = \frac{1}{\Delta\kappa_x\Delta\kappa_y} \left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M} j(n,m) e^{-i(\kappa_{x_p} n\Delta x+\kappa_{y_q} m\Delta y)}\right]$$
$$\times \left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M} k(n,m) e^{i(\kappa_{x_p} n\Delta x+\kappa_{y_q} m\Delta y)}\right] . \qquad (7.22)$$

If j = k, Eq. 7.22 reduces to

$$S_{jj}(\kappa_{x_p},\kappa_{y_q}) = \frac{1}{\Delta\kappa_x\Delta\kappa_y}\left|\frac{1}{NM}\, j(n,m)e^{-i(\kappa_{x_p}n\Delta x+\kappa_{y_q}m\Delta y)}\right|^2 \qquad (7.23)$$

By utilizing Eqs. 7.22 and 7.23 together with the FFT technique, the power spectral density function $S_{jk}(\kappa_x,\kappa_y)$ can be efficiently estimated from a set of discrete data $j(n,m)$ equally spaced $\Delta x = L_x/N$, $\Delta y = L_y/M$ in the region $0 \leq x \leq L_x$ and $0 \leq y \leq L_y$.

8. SUMMARY

A new version of the simulation equations for bi-variate two-dimensional stochastic processes is described which is consistent with the spectral representation of homogeneous stochastic processes. The new version is given by Eqs. 2.17 and 2.19 for quadrant symmetry. Also, the characterization of bi-variate spatially two-dimensional time-space stochastic processes is presented with a numerical example based on the analysis of seismic array records in Taiwan (SMART-1). Finally, the essentials for estimating the power spectral density function of bi-variate two-dimensional stochastic processes from a set of measured data in finite regions are presented.

For simplicity in this study, we discuss bi-variate one-dimensional processes, bi-variate two-dimensional processes and bi-variate spatially two-dimensional time-space processes. However, the results may be easily extended to multi-variate multi-dimensional processes by following the same procedures as those used in this study.

9. ACKNOWLEDGEMENTS

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The views, opinions and findings expressed in this study are those of the

authors and should not be construed as representing an official position, policy or decision on the part of the organization mentioned above.

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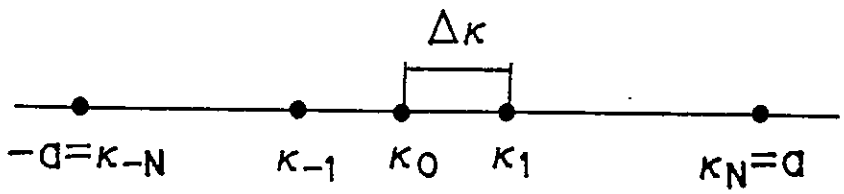


Fig. 2-1 Discretization of Wave Number Domain

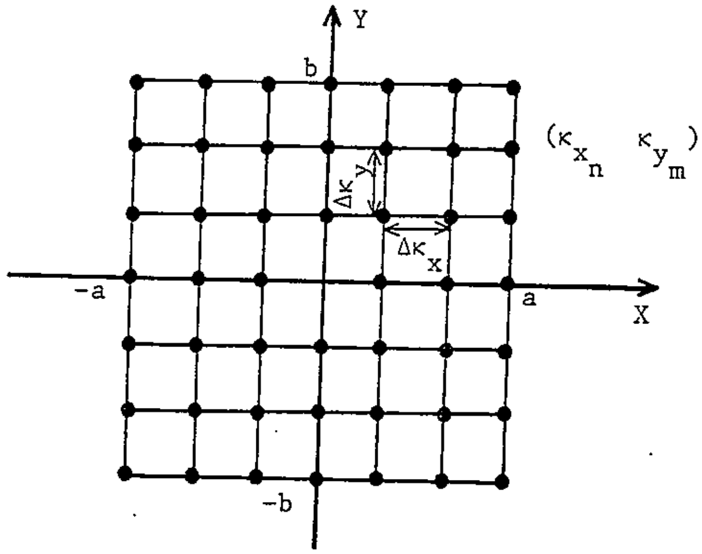
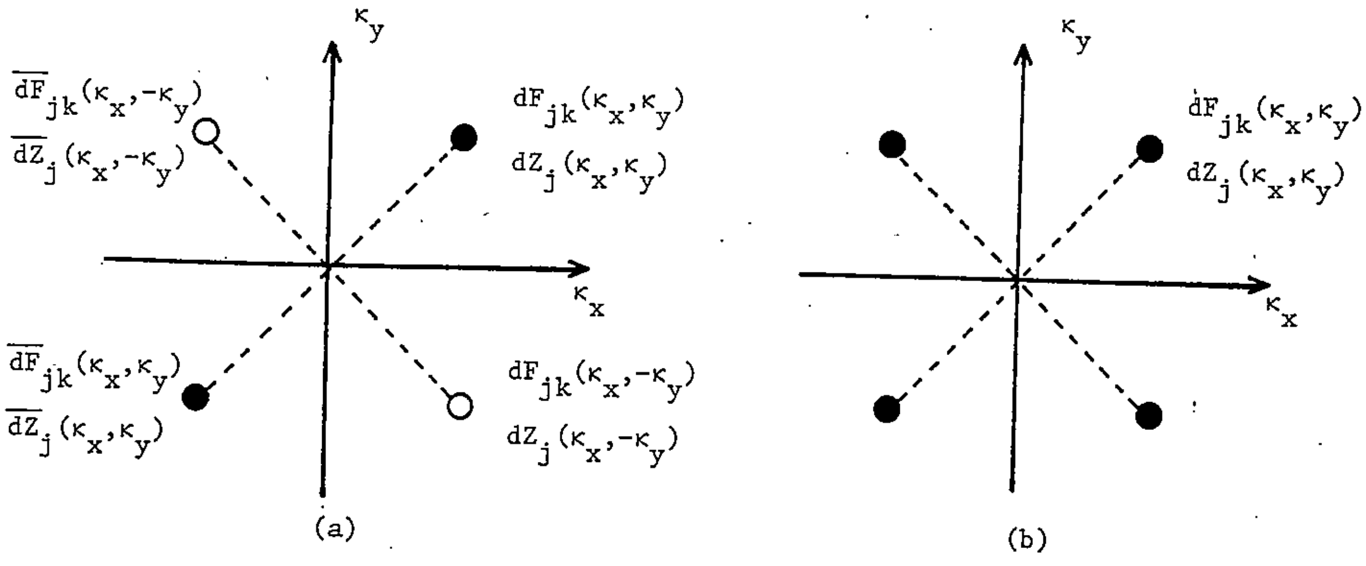


Fig. 3-1 Discretization of x-y Plane



(Homogeneous Two-Dimensional Case)

(Quadrant Symmetry Case)

Fig. 3-2 Characteristics of Homogeneous Two-Dimensional Stochastic Fields in Wave-Number Domain

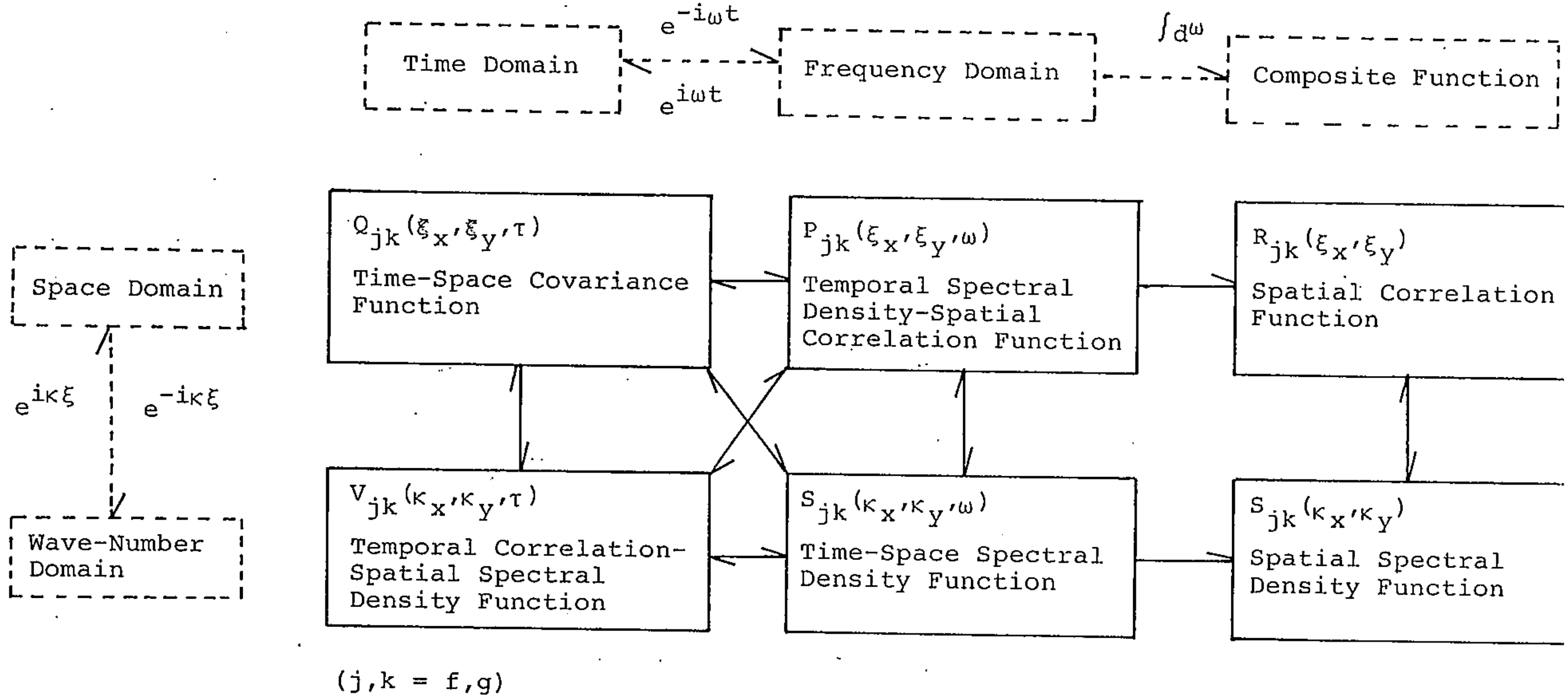


Fig. 5-1 Relationship among The Various Second Moments for Stationary-Homogeneous Time-Space Stochastic Fields

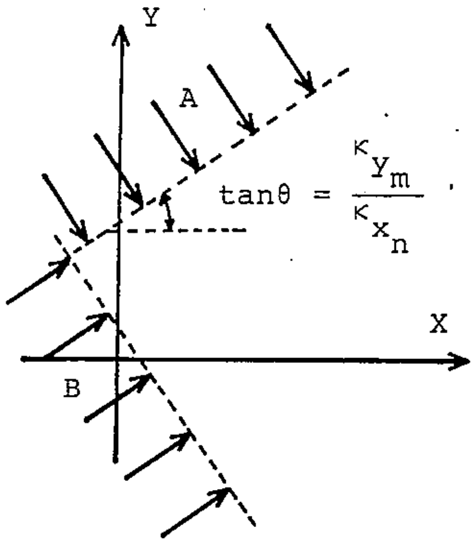
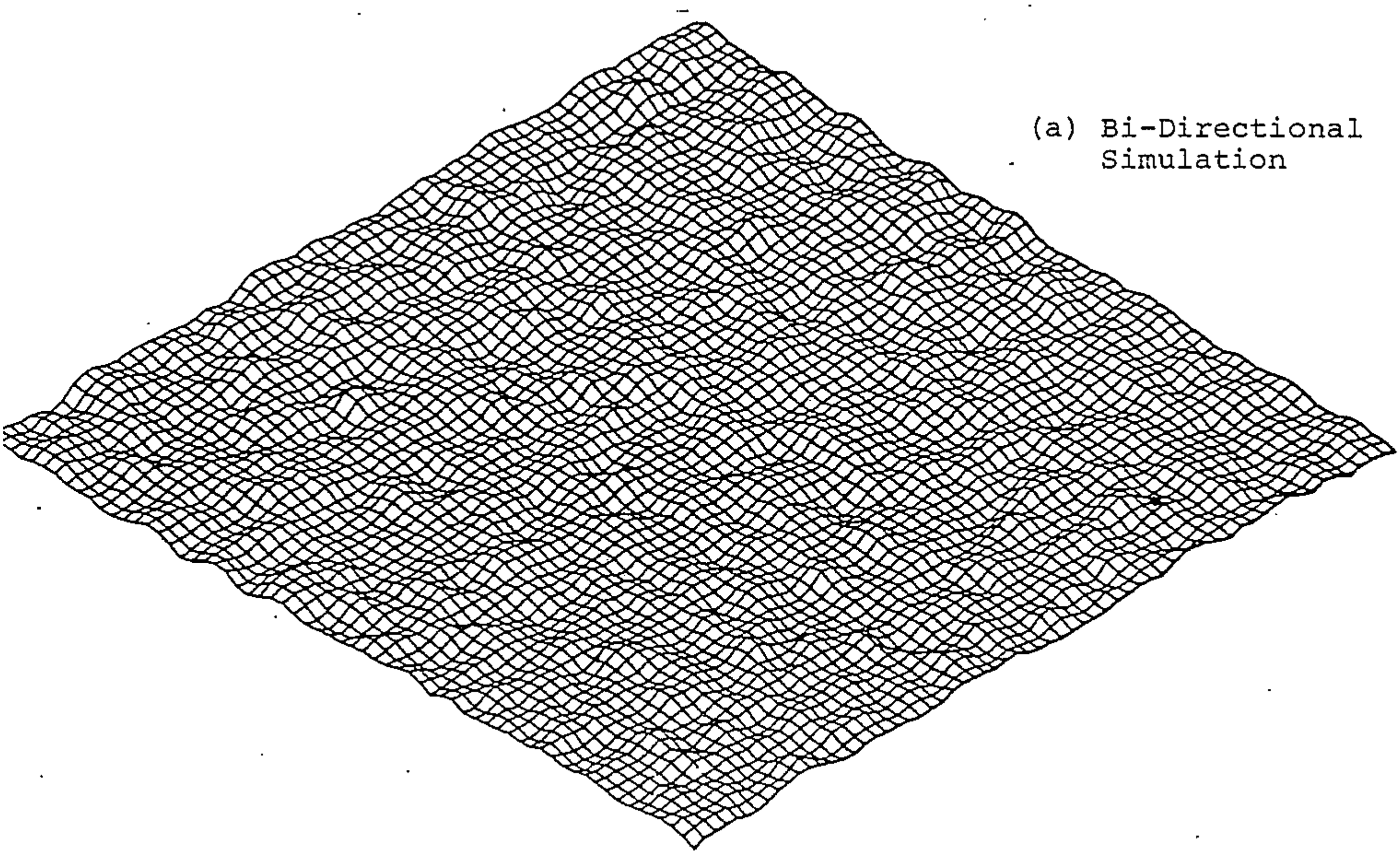
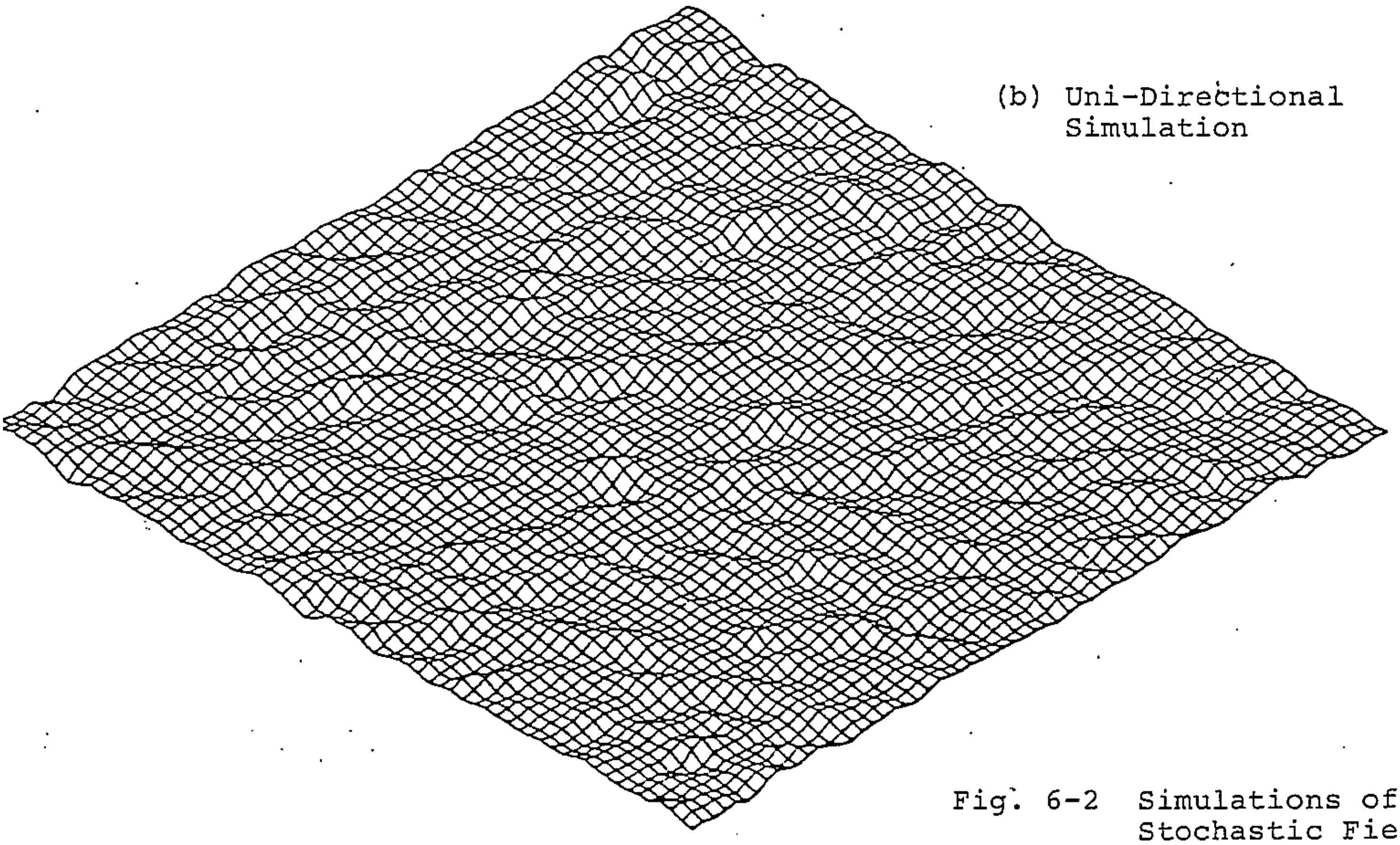


Fig. 6-1 Bi-Directional Waves in X-Y Plane



(a) Bi-Directional Simulation



(b) Uni-Directional Simulation

Fig. 6-2 Simulations of Stochastic Field

$$R_{ff}(\xi) = \sigma_0^2 e^{-\xi^2/d^2}$$

$$S_{ff}(\kappa) = \sigma_0^2 \frac{d^2}{4\pi} e^{-d^2\kappa^2/4}$$

$$\xi = \sqrt{\xi_x^2 + \xi_y^2}$$

$$\kappa = \sqrt{\kappa_x^2 + \kappa_y^2}$$

$$d = 1.0$$

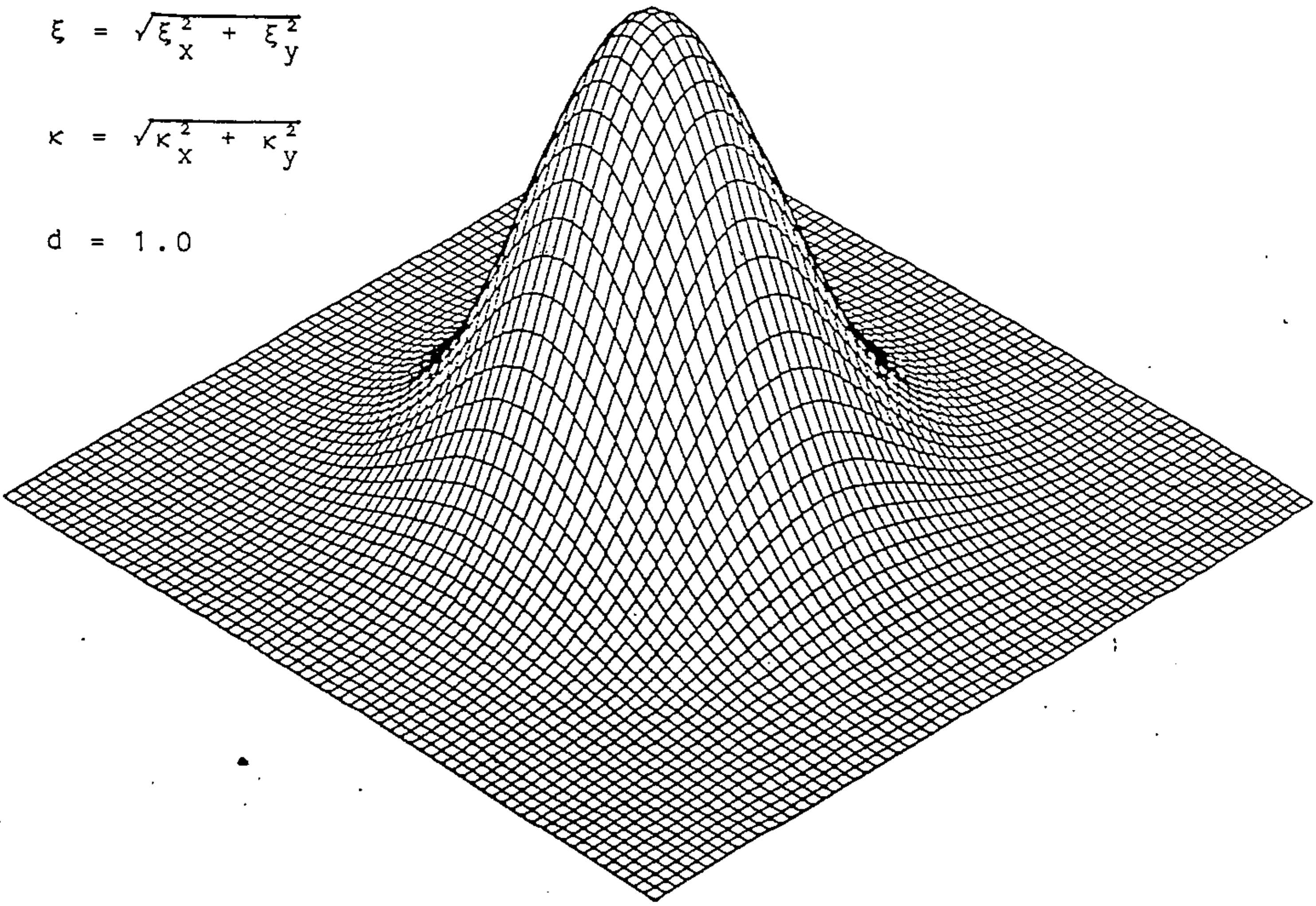


Fig. 6-3 Isotropic Spectral Density Function

DOUBLE SYMMETRIC SPECTRUM

OR QUADRANT SPECTRUM

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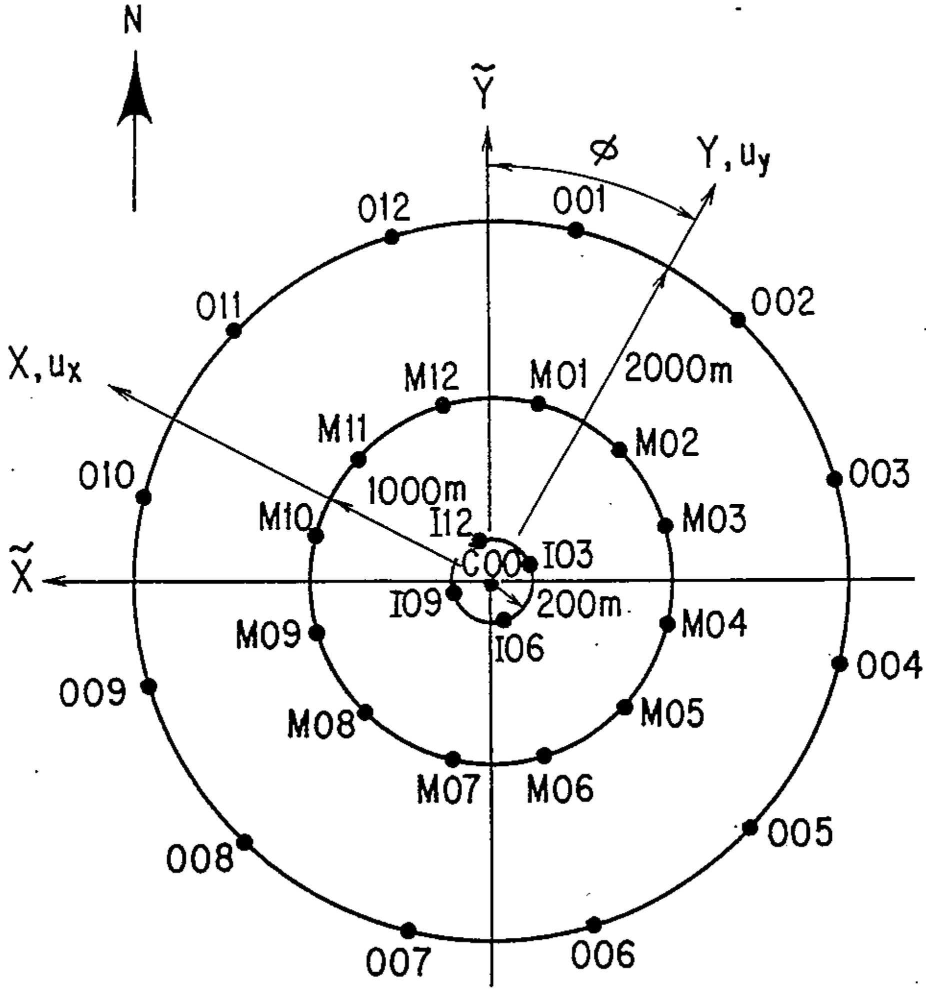

Fig. 6-4 THE SMART-1 ARRAY AND COORDINATE SYSTEM

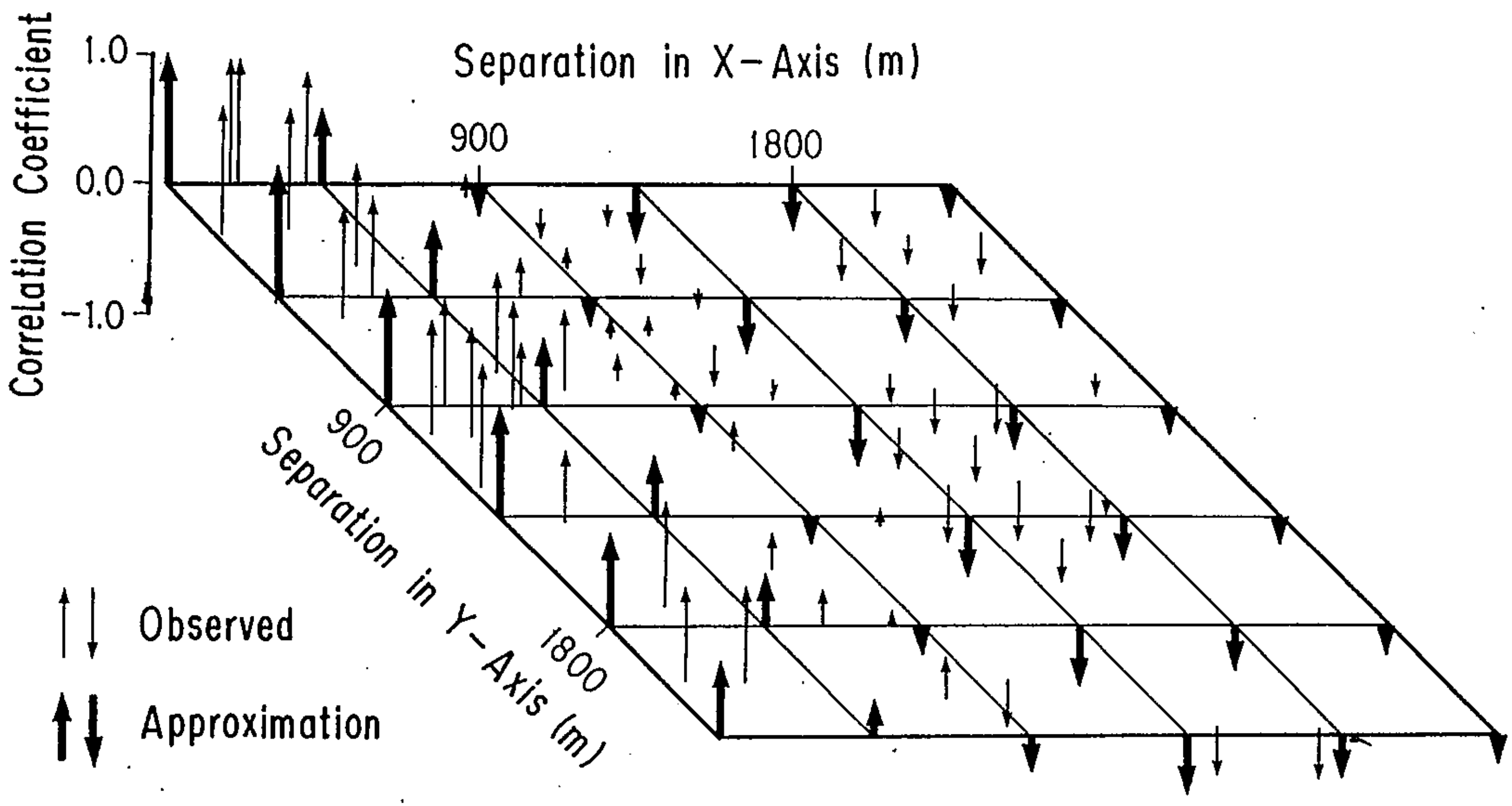

Fig. 6-5 OBSERVED AND APPROXIMATED 2-D SPATIAL CORRELATIONS FOR RUX (EVENT 5)

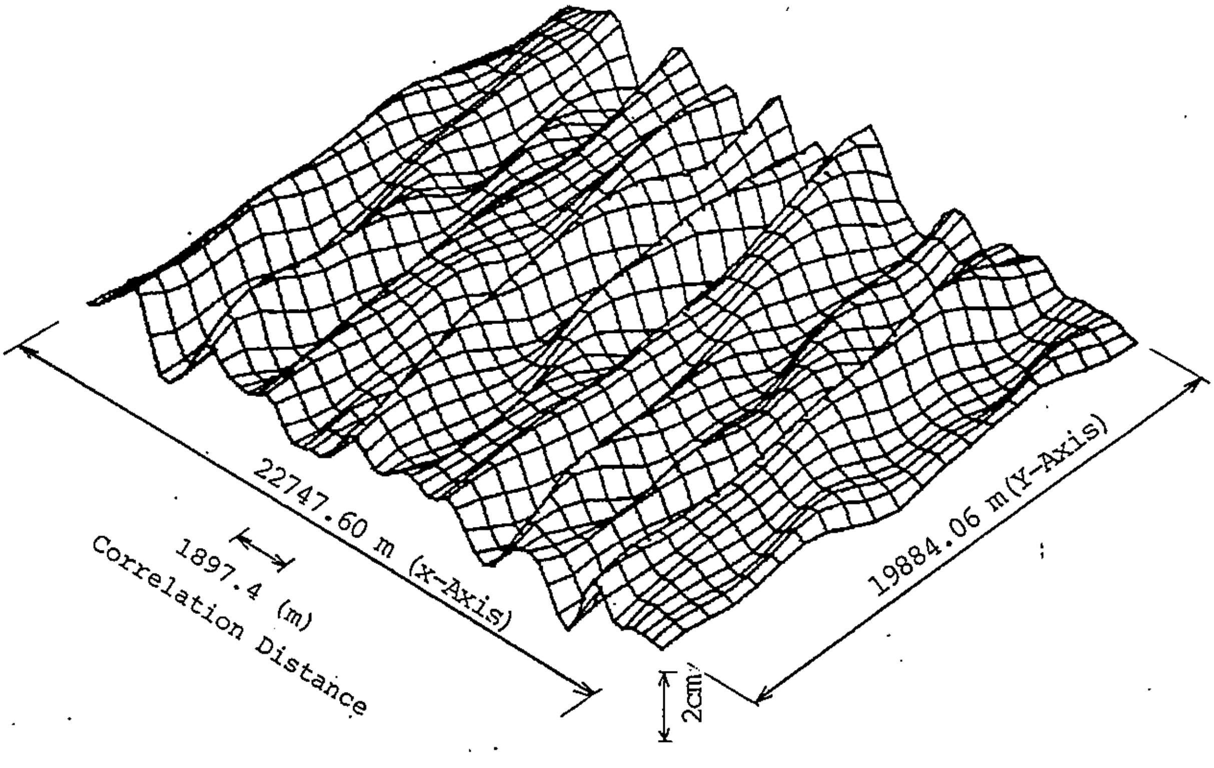

Fig. 6-8 Sample Function of f(x,y) for Case 3