A Note on Almost Dedekind Domains

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Throughout this discussion, R will be an integral domain. A. K. Tiwary showed in [3] that, if Rp is PID for a maximal ideal P of R, the injective hull $E=E(R/P)$ of R/P is isomorphic to any of non-zero homomorphic images of E. We shall show in this paper that a local domain with this property is PID. Moreover, we shall show that a locally noetherian domain with the property mentioned above must be an almost Dedekind domain.

We denote by $E(M)$ the injective hull of an R-module M. Let E be an injective R-module, N be a submodule of E and A be an ideal of R; we put $A^*=\{x\in E\mid ax=0$ for every a$\in A\}$ and $N^*=\{r\in R\mid rx=0$ for every $x\in N\}$. If R is a local ring, $\bar{R}$ denotes the completion of R. When R is a quasi-local domain with the maximal ideal P and $E=E(R/P)$, we define a homomorphism ϕ_A: $E\to\overset{n}{\oplus}E$ by $\phi_A(x)=(a_1x,\dots,\ a_nx)$ for an ideal $A=(a_1,\dots,\ a_n)$ of R. ($\overset{n}{\oplus}E$ denotes a direct sum of n copies of E.) We denotes Imϕ_A by E_A. Since Ker $\phi_A=A^*$, E_A is independent of the choice of the ideal basis of A up to isomorphisms.

Lemma. *Let R be a local domain with the maximal ideal P, and set $E=E(R/P)$. Then for every ideal A of R, we have*

(1) $A^{**}=A$.

(2) $A^*=(A\bar{R})^*$ *as an $\bar{R}$-module.*

Proof. (1) This property is well-known. (cf. [1], [2]) (2) By Corollary of Proposition 2 of [4], A^* has the structure of an $\bar{R}$-module. The result (2) follows immediately.

With this preparation, we have

Theorem 1. *Let R be a local domain with the maximal ideal P and set $E=E(R/P)$. Then R is PID if and only if E is isomorphic to any of non-zero homomorphic images of E.*

Proof. $\Rightarrow$This follows from [3].

$\Leftarrow$Suppose that A is non-zero ideal of R. Then E_A is isomorphic to E/A^*. If $E=A^*$, by Lemma, $E=(A\bar{R})^*$ as an $\bar{R}$-module. Then by Theorem 4.2 of [2] and the above Lemma, $A\bar{R}=(A\bar{R})^{**}=E^*=0$; i. e. $A=0$. This is a contradiction. Hence, $E_A\neq 0$. From the assumption and Corollary of Proposition 5 of [4], it follows that A is principal. Thus, the proof is complete.

A. K. Tiwary remarked in [3] that the indecomposable torsion modules over a Dedekind domain all have the property that they are isomorphic to any their non-zero homorphic images. The following result contains this fact.

Theorem 2. *Let R be a locally noetherian domain. Then R is an almost Dedekind domain if and only if, for each prime ideal P of R, $E(R/P)$ is isomorphic to any of non-zero homomorphic images of $E(R/P)$.*

Proof. $\Rightarrow$This follows from Proposition 2 of [4] and Theorem 1.
$\Leftarrow$Let P be an arbitrary maximal ideal of R and set $E=E(R/P)$. Then E is an injective hull of R_p/PR_p as an R_p-module by Theorem 3.6 of [2] (cf. [3]). For each non-zero R_p-submodule N of E, it follows from the assumption that E is isomorphic to E/N as an R-module. Also, $\mathrm{Hom}_R(E, E/N)=\mathrm{Hom}_{R_p}(E, E/N)$, since E and N have the structures of R-and R_p-modules. Therefore, E is isomorphic to E/N as an R_p-module. From Theorem 1, R_p is PID; i.e. R is an almost Dedekind domain. Thus, the proof is complete.

References

1) T. Ishikawa, Faithfully exact functors and their applications to projective modules and injective modules, Nagoya Math. J. **24** (1964), 29-42.
2) E. Matis, Injective modules over noetherian rings, Pacific J. Math. **8** (1958), 511-528.
3) A.K. Tiwary, On the qutients of indecomposable injective modules, Canadian Math. Bull. **9** (1966), 187-190.
4) H. Uda, On a characterization of almost Dedekind domains, Hiroshima Math. J. **2** (1972), 339-344.

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