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The additive structure of $\widetilde{KO}(S^{4n+3}/Q_t)$

Dedicated to Professor Masahiro Sugawara on his 60th birthday

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§ 1. Introduction

Let t be a positive integer and let Q_t be the group of order $4t$ given by

$$Q_t = \{ x, y : x^t = y^2, xyx = y \},$$

the group generated by two elements x and y with the relations $x^t = y^2$ and $xyx = y$, that is, Q_t is the subgroup of the unit sphere S^3 in the quaternion field H generated by the two elements

$$x = \exp(\pi i/t) \text{ and } y = j;$$

and $Q_1 = Z_4$ and Q_t for $t = 2^{m-1}(m \geqq 2)$ is the generalized quaternion group which is denoted by H_m in [6] and [7].

Then, Q_t acts on the unit sphere S^{4n+3} in the quaternion $(n+1)$-space H^{n+1} by the diagonal action, and we have the quotient manifold

$$S^{4n+3}/Q_t \text{ of dimension } 4n+3.$$

Some partial results on the reduced KO-ring $\widetilde{KO}(S^{4n+3}/Q_t)$ of this manifold are obtained by [7], D. Pitt [17], H. Ōshima [16], [15] and T. Kobayashi [13]. Recently, T. Kobayashi has determined the additive structure of $\widetilde{KO}(S^{4n+3}/Q_4)$ in [14]. In this paper, we shall determine completely the additive structure of $\widetilde{KO}(S^{4n+3}/Q_t)$.

Throughout this paper, we identify the orthogonal representation ring $RO(Q_t)$ with the subring $c(RO(Q_t))$ of the unitary representation ring $R(Q_t)$ through the complexification $c: RO(Q_t) \longrightarrow R(Q_t)$, since c is a ring monomorphism (cf. (2.1)).

Consider the complex representations a_0, a_1, a_2 and b_1 of Q_t given by

$$\begin{cases} a_0(x) = 1, \\ a_0(y) = -1, \end{cases} \quad \begin{cases} a_i(x) = -1, \\ a_i(y) = \begin{cases} (-1)^{i-1} i & \text{if } t \text{ is odd,} \\ (-1)^{i-1} & \text{if } t \text{ is even,} \end{cases} \end{cases} \quad \begin{cases} b_1(x) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \\ b_1(y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{cases}$$

Then

$$a_i - 1,\ 2(b_1 - 2),\ (b_1 - 2)^2 \in \widetilde{RO}(Q_t) \qquad \text{(cf. Prop.2.7),}$$

where $\widetilde{RO}(Q_t)$ is the reduced orthogonal representation ring.

Consider the elements

$$\text{(1.1)} \qquad \alpha_i = \xi(a_i - 1),\quad 2\beta_1 = \xi(2b_1 - 4),\quad \beta_1^2 = \xi((b_1 - 2)^2)$$

in $\widetilde{KO}(S^{4n+3}/Q_t)$ (cf. (3.3)), where $\xi: \widetilde{RO}(Q_t) \longrightarrow \widetilde{KO}(S^{4n+3}/Q_t)$ is the natural ring homomorphism (cf. (3.1)). Furthermore, consider the folllowing subgroups of Q_t:

(1.2) $G_0 = Q_r$ generated by x^q and y, $G_1 = Z_q$ generated by x^{2r},

where $t = rq$, $r = 2^{m-1}$, $m \geqq 1$ and q is odd. Then, we have the ring homomorphisms

$$
\begin{aligned}
&i_0^* : \widetilde{KO}(S^{4n+3}/Q_t) \longrightarrow \widetilde{KO}(S^{4n+3}/Q_r),\\
(1.3)\qquad &i_1^* : \widetilde{KO}(S^{4n+3}/Q_t) \longrightarrow \widetilde{KO}(L^{2n+1}(q)) \ (L^{2n+1}(q) = S^{4n+3}/Z_q),\\
&i^* : \widetilde{KO}(L^{2n+1}(q)) \longrightarrow \widetilde{KO}(L_0^{2n+1}(q)),
\end{aligned}
$$

induced from the natural projections $i_k : S^{4n+3}/G_k \longrightarrow S^{4n+3}/Q_t$ and the inclusion $i : L_0^{2n+1}(q) \longrightarrow L^{2n+1}(q)$, where $L_0^{2n+1}(q)$ is the $(4n+2)$-skeleton of $L^{2n+1}(q)$ the standard lens space modulo q.

Then, we have the following

THEOREM 1.4. (i) *The ring $\widetilde{KO}(S^{4n+3}/Q_t)$ is generated by the elements α_0, $\alpha_1 + \alpha_2$ if $t = 1$, α_0, $\alpha_1 + \alpha_2$, $2\beta_1$ and β_1^2 if $t \geqq 3$ is odd, α_0, α_1, $2\beta_1$ and β_1^2 if t is even, respectively, where α_i, $2\beta_1$ and β_1^2 are the elements in* (1.1).

(ii) *Put $t = rq$ where $r = 2^{m-1}$, $m \geqq 1$ and q is odd. Then, we have the ring isomorphism*

$$\pi = i_0^* \oplus i^* i_1^* : \widetilde{KO}(S^{4n+3}/Q_t) \cong \widetilde{KO}(S^{4n+3}/Q_r) \oplus \widetilde{KO}(L_0^{2n+1}(q)),$$

where i_0^*, i_1^* and i^* are the ones in (1.3). *Further, there hold the equalities*

$$
\begin{cases}
\pi(\alpha_0) = \alpha_0,\ \pi(\alpha_1 + \alpha_2) = \alpha_1 + \alpha_2, & \\
\pi(2\beta_1) = 2\alpha_1 + 2\alpha_2 + 2\bar{\sigma}, & \text{if } t \text{ is odd},\\
\pi(\beta_1^2) = -4\alpha_1^3 - 10\alpha_1^2 - 12\alpha_1 + \bar{\sigma}^2, &
\end{cases}
$$

$$
\begin{cases}
\pi(\alpha_i) = \alpha_i \quad (i = 0, 1, 2), & \\
\pi(2\beta_1) = 2\beta_1 + 2\bar{\sigma}, & \text{if } t \text{ is even},\\
\pi(\beta_1^2) = \beta_1^2 + \bar{\sigma}, &
\end{cases}
$$

where $\bar{\sigma}$ is the real restriction of the stable class $\eta - 1$ of the canonical complex line bundle η over $L_0^{2n+1}(q)$ and it generates the ring $\widetilde{KO}(L_0^{2n+1}(q))$ (cf. [11, Prop. 2.11]), *and the additive structure of $\widetilde{KO}(L_0^{2n+1}(q))$ is given in* [9, Th.1.10 and (6.1)].

Consider the following integers $\bar{u}(i)$ and the elements $\bar{\delta}_i$ and $\bar{\alpha}_1$ in $\widetilde{KO}(S^{4n+3}/Q_r)$ with $r = 2^{m-1}$ $(m \geqq 2)$, where α_i and $2\beta_1$ are the ones in (1.1) for $t = r$ and

$$2\beta(0) = 2\beta_1 \text{ and } \beta(s) = \beta(s-1)^2 + 4\beta(s-1) \quad (s \geqq 1):$$

For $i = 2^s + d \leqq N' = \min\{r, n\}$ with $0 \leqq s < m$ and $0 \leqq d < 2^s$, put

$$n' = 2n+1 \text{ if } n \text{ is odd}, = 2n \text{ if } n \text{ is even},$$

$$n' = 2^s a_s' + b_s', \qquad 0 \leqq b_s' < 2^s;$$

(1.5) $\bar{u}(1) = 2^{m-2+a'_0}$, $\bar{\delta}_1 = 2\beta_1$ if $i = 1$;

$$
\left\{
\begin{array}{l}
\bar{u}(2) = \begin{cases} 2^{m-3+a'_1} & (n:\text{odd}), \\ 2^{m-2+a'_1} & (n:\text{even}), \end{cases} \\
\bar{\delta}_2 = \begin{cases} \beta(1) - 2^{1+a'_1}\beta(0) - R_0(1, 0;\ a'_1+1) & (n:\text{odd}), \\ \beta(1) & (n:\text{even}), \end{cases}
\end{array}
\right.
\qquad \text{if } i = 2;
$$

$$
\left\{
\begin{array}{l}
\bar{u}(i) = 2^{m-s-2+a'_s}, \\
\bar{\delta}_i = \begin{cases} \sum_{t=0}^{s}(-1)^{2^t+1} 2^{(2^t-1)(a'_s+1)}\beta(s-t) - R_0(s, 0;\ a'_s+1) & (n:\text{odd}), \\ \sum_{t=0}^{s} 2^{(2^t-1)(a'_s+1)}\beta(s-t) & (n:\text{even}), \end{cases}
\end{array}
\right.
$$

$$
\text{if } i = 2^s \ (2 \leqq s \leqq m-1);
$$

$$
\left\{
\begin{array}{l}
\bar{u}(i) = 2^{m-s-3+a(i)}, \\
\bar{\delta}_i = \begin{cases} 2\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t)) + \sum_{t=0}^{s}(-1)^{2^t} 2^{(2^{t+1}-1)a(i)}\beta_1^d\beta(s-t) & (d:\text{odd}), \\ \beta_1^{d-2}\beta(2)\prod_{t=1}^{s-1}(2+\beta(t)) + R(s, d;\ a(i)) & (n:\text{odd},\ d:\text{even}), \\ \beta_1^{d-2}\beta(2)\prod_{t=1}^{s-1}(2+\beta(t)) + \sum_{t=0}^{s}(-1)^{2^t+a(i)} 2^{(2^{t+1}-1)a(i)-1}\beta_1^d\beta(s-t) & (n:\text{even},\ d:\text{even}), \end{cases}
\end{array}
\right.
$$

$$
a(i) = \begin{cases} a'_{s+1}+1 & \text{for } 2d \leqq b'_{s+1}, \\ a'_{s+1} & \text{for } 2d > b'_{s+1}, \end{cases}
$$

$$
\text{if } i = 2^s + d \geqq 3,\ d \geqq 1;
$$

$$
\bar{\alpha}_1 = \begin{cases} \alpha_1 & (n:\text{even or } m=2), \\ \alpha_1 \pm 2^{m-2+n}\beta_1 & (n:\text{odd and } m \geqq 3), \end{cases}
$$

where $R_0(s, d, a'_s+1)$ and $R(s, d;\ a(i))$ are the ones in Propositions 7.1 and 7.2, respectively.

Then, the additive structure of $\widetilde{KO}(S^{4n+3}/Q_r)$ is given by the following theorem, where $Z_k\langle x\rangle$ denotes the cyclic group of order k generated by x:

THEOREM 1.6. *Let $r = 2^{m-1}$, $m \geqq 2$ and $N' = \min\{r, n\}$. Then, we have*

$$
\widetilde{KO}(S^{4n+3}/Q_r) = \begin{cases} Z_{2^{n+1}}\langle \alpha_0\rangle \oplus Z_{2^{n+1}}\langle \bar{\alpha}_1\rangle \oplus \sum_{i=1}^{N'} Z_{\bar{u}(i)}\langle \bar{\delta}_i\rangle & (n:\textit{odd}), \\ Z_{2^{n+2}}\langle \alpha_0\rangle \oplus Z_{2^{n+2}}\langle \bar{\alpha}_1\rangle \oplus \sum_{i=1}^{N'} Z_{\bar{u}(i)}\langle \bar{\delta}_i\rangle & (n:\textit{even}). \end{cases}
$$

We notice that the additive structure of $\widetilde{KO}(S^{4n+3}/Q_1) = \widetilde{KO}(L^{2n+1}(4))$ is determined in [12, Th.B].

We prepare some results on the complex and orthogonal representation rings $R(Q_t)$,

$RO(Q_t)$, $R(G_k)$ and $RO(G_k)$ for Q_t and the subgroups G_k given in (1.2), and the symplectic representation group $RSp(Q_r)$ $(r=2^{m-1})$ in §2.

In §3, we define the elements α_i $(i=0, 1, 2)$, $2\beta_{2j+1}$ and β_{2j} of $\widetilde{KO}(S^{4n+3}/Q_t)$ and study the homomorphisms $i_k^*: \widetilde{KO}(S^{4n+3}/Q_t) \longrightarrow \widetilde{KO}(S^{4n+3}/G_k)$ and $i^*: \widetilde{KO}(L^{2n+1}(q)) \longrightarrow \widetilde{KO}(L_0^{2n+1}(q))$ of (1.3) in Lemma 3.6, Propositions 3.8 and 3.12. Also, the fundamental relations in $\widetilde{KO}(S^{4n+3}/Q_r)$ $(r=2^{m-1})$ are given in Lemma 3.14, which play the important parts in the subsequent sections.

In §4, we first estimate an upper bound of the order of $\widetilde{KO}(S^{4n+3}/Q_t)$ by using the Atiyah–Hirzebruch spectral sequence, and especially we determine the order of $\widetilde{KO}(S^{4n+3}/Q_r)$ $(r=2^{m-1})$ in Proposition 4.13. Furthermore, we prove Theorem 1.4 in Theorem 4.15 and Remark 4.16 by using the known results about the order of $\widetilde{KO}(L_0^{2n+1}(q))$ given in [11, Prop.2.11], the order of $\widetilde{KO}(S^{4n+3}/Q_r)$ $(r=2^{m-1})$ given in Proposition 4.13 and the results obtained in §3.

In §5 (resp. §8), we give some relations in $\widetilde{KO}(S^{4n+3}/Q_r)$ $(r=2^{m-1})$ for odd n (resp. even n), which are useful in the next section.

In §6 (resp. §9), we prove some basic relations concerned with an additive base of $\widetilde{KO}(S^{4n+3}/Q_r)$ $(r=2^{m-1})$ for odd n (resp. even n) by making use of the relations given in §5 (resp. §8)

In §7 (resp. §10), Theorem 1.6 for odd n (resp. even n) is proved by combining the results given in the previous sections. Also, as the corollary of Theorem 1.6, we have the order of $\bar{\delta}_1$ in $\widetilde{KO}(S^{4n+3}/Q_r)$ $(r=2^{m-1})$, which is already proved in [13, Cor. 1.7].

§2. The representation rings of Q_t

We denote the unitary (resp. orthogonal) representation ring of the group G by $R(G)$ (resp. $RO(G)$) and the symplectic representation group by $RSp(G)$. By the natural inclusion

$$O(n) \subset U(n),\ U(n) \subset O(2n),\ Sp(n) \subset U(2n) \text{ and } U(n) \subset Sp(n),$$

the following group homomorphisms are defined:

$$RO(G) \underset{c}{\overset{r}{\leftrightarrows}} R(G) \underset{h}{\overset{c'}{\leftrightarrows}} RSp(G).$$

The following facts (2.1) are well known (cf. eg. [2]).

(2.1) *These representation groups are free, and* c *is a ring homomorphism. Also*

$$rc=2,\ hc'=2,\ cr=1+t=c'h,$$

(t *denotes the conjugation*), *and* c *and* c' *are monomorphic.*

Hence throughout this paper, we identify

$$RO(G) \text{ with } c(RO(G)), \text{ and } RSp(G) \text{ with } c'(RSp(G)).$$

Let t be a positive integer and let Q_t be the subgroup of order $4t$ of the unit sphere S^3 in the quaternion field H generated by the two elements

$$x = \exp(\pi \boldsymbol{i}/t) \quad \text{and} \quad y = \boldsymbol{j}.$$

Consider the complex representations a_i $(i=0, 1, 2)$ and b_j $(j \in Z)$ of Q_t given by

$$(2.2) \qquad \begin{cases} a_0(x) = 1, \\ a_0(y) = -1, \end{cases} \quad \begin{cases} a_i(x) = -1, \\ a_i(y) = \begin{cases} (-1)^{i-1}\boldsymbol{i} & \text{if } t \text{ is odd}, \\ (-1)^{i-1} & \text{if } t \text{ is even}, \end{cases} \end{cases} \quad \begin{cases} b_j(x) = \begin{pmatrix} x^j & 0 \\ 0 & x^{-j} \end{pmatrix}, \\ b_j(y) = \begin{pmatrix} 0 & (-1)^j \\ 1 & 0 \end{pmatrix}. \end{cases}$$

Then, we see easily the following

PROPOSITION 2.3. (cf. [4, §47.15, Example 2]). *The complex representation ring $R(Q_t)$ is a free Z-module with basis* 1, a_i $(i=0, 1, 2)$ *and* b_j $(1 \leqq j < t)$ *and multiplicative structure is given as follows*:

$$a_0^2 = 1, \quad a_1^2 = \begin{cases} a_0 & \text{if } t \text{ is odd}, \\ 1 & \text{if } t \text{ is even}, \end{cases} \qquad a_2 = a_0 a_1, \quad b_0 = 1 + a_0, \quad b_t = a_1 + a_2,$$

$$b_{t+i} = b_{t-i}, \quad b_{-i} = b_i, \quad b_i b_j = b_{i+j} + b_{i-j}, \quad a_0 b_i = b_i, \quad a_1 b_i = b_{t-i},$$

Let

$$(2.4) \qquad \alpha_i = a_i - 1 \ (i = 0, 1, 2) \quad \text{and} \quad \beta_j = b_j - 2 \ (j \in Z)$$

be the elements in the reduced representation ring $\widetilde{R}(Q_t)$. Then, we have

PROPOSITION 2.5. (cf. [6, Prop.3.3]) *The reduced representation ring $\widetilde{R}(Q_t)$ is a free Z-module with basis* α_i $(i=0, 1, 2)$ *and* β_j $(1 \leqq j < t)$, *and multiplicative structure is given as follows*:

$$\alpha_0^2 = -2\alpha_0, \quad \alpha_1^2 = \begin{cases} \alpha_0 - 2\alpha_1 & \text{if } t \text{ is odd}, \\ -2\alpha_1 & \text{if } t \text{ is even}, \end{cases} \qquad \alpha_2 = \alpha_0\alpha_1 + \alpha_0 + \alpha_1, \quad \beta_0 = \alpha_0,$$

$$\beta_t = \alpha_1 + \alpha_2, \quad \beta_{t+i} = \beta_{t-i}, \quad \beta_{-i} = \beta_i,$$

$$\beta_i \beta_j = \beta_{i+j} + \beta_{i-j} - 2(\beta_i + \beta_j), \quad \alpha_0 \beta_i = -2\alpha_0, \quad \alpha_1 \beta_i = \beta_{t-i} - \beta_i - 2\alpha_1.$$

These show that the ring $\widetilde{R}(Q_t)$ is generated by α_1 if $t=1$, α_1 and β_1 if $t \geqq 3$ is odd, and α_0, α_1 and β_1 if t is even.

Regarding $RO(Q_t)$ as the subring of $R(Q_t)$ under $c: RO(Q_t) \longrightarrow R(Q_t)$ in (2.1), we have

PROPOSITION 2.6 (cf. [5, (3.5) and (12.3)]). *$RO(Q_t)$ is a free Z-module with basis* 1, a_0, $a_1 + a_2$, b_{2j} *and* $2b_{2j+1}$ $(1 \leqq 2j,\ 2j+1 < t)$ *if t is odd, and* $1, a_i$ $(i=0, 1, 2)$, b_{2j} *and* $2b_{2j+1}$ $(1 \leqq 2j,\ 2j+1 < t)$ *if t is even.*

From (2.4), Propositions 2.5 and 2.6, we have

PROPOSITION 2.7. *The reduced representation ring* $\widetilde{RO}(Q_t)$ *is a free* Z*-module with basis* α_0, $\alpha_1 + \alpha_2$, β_{2j} *and* $2\beta_{2j+1}$ $(1 \leqq 2j,\ 2j+1 < t)$ *if* t *is odd, and* α_i $(i=0, 1, 2)$, β_{2j} *and* $2\beta_{2j+1}$ $(1 \leqq 2j,\ 2j+1 < t)$ *if* t *is even. These show that the ring* $\widetilde{RO}(Q_t)$ *is generated by* α_0, $\alpha_1 + \alpha_2$ *if* $t=1$, α_0, $\alpha_1 + \alpha_2$, $2\beta_1$ *and* β_1^2 *if* $t \geqq 3$ *is odd,* α_0, α_1, $2\beta_1$ *and* β_1^2 *if* t *is even.*

Regarding $RSp(Q_r)$ $(r=2^{m-1})$ as the subgroup under $c' : RSp(Q_r) \longrightarrow R(Q_r)$ in (2.1), we have

PROPOSITION 2.8. (cf. [17, Prop.1.6]). $RSp(Q_r)$ $(r=2^{m-1})$ *is a free* Z*-module with basis* 2, $2a_i$ $(i=0, 1, 2)$, $2b_{2j}$ *and* b_{2j+1} $(1 \leqq 2j,\ 2j+1 < r)$.

The following lemmas are well known:

LEMMA 2.9 (cf. [1, §8]). $R(Z_k)$ *is the truncated polynomial ring* $Z[\mu]/\langle \mu^k - 1 \rangle$, *where* μ *is given by* $z \longrightarrow \exp(2\pi \boldsymbol{i}/k)$ *for the generator* z *of* Z_k *and* $\langle \mu^k - 1 \rangle$ *means the ideal of* $Z[\mu]$ *generated by* $\mu^k - 1$.

LEMMA 2.10 (cf. [5, (3.5) and (12.3)]). *The ring* $\widetilde{RO}(Z_k)$ *is generated by* $r(\mu-1)$ *if* k *is odd,* $\rho - 1$ *and* $r(\mu - 1)$ *if* k *is even, where* r *is the real restriction and* ρ *is a real representation given by* $z \longrightarrow -1$ *for the generator* z *of* Z_k.

Consider the following subgroup G_k of Q_t, where $t = rq$, $r=2^{m-1}$, $m \geqq 1$ and q is odd:

(2.11) $\qquad G_0 = Q_r$ generated by x^q and y, $G_1 = Z_q$ generated by x^{2r}.

Then the inclusion $i_k : G_k \subset Q_t$ induces the ring homomorphism

$$i_k^* : \widetilde{RO}(Q_t) \longrightarrow \widetilde{RO}(G_k) \tag{2.12}$$

by the restriction of representations of Q_t to G_k.

By [9, Prop. 2.9], Proposition 2.7 and Lemma 2.10, we see easily the following

LEMMA 2.13. (i) i_0^* *is an epimorphism and*

$$\begin{cases} i_0^*(\alpha_0) = \alpha_0, \quad i_0^*(\alpha_1 + \alpha_2) = \alpha_1 + \alpha_2, \\ i_0^*(\beta_2) = \alpha_0, \quad i_0^*(2\beta_{2i+1}) = 2(\alpha_1 + \alpha_2), \end{cases} \qquad \textit{if } t \textit{ is odd,}$$

$$\begin{cases} i_0^*(\alpha_i) = \alpha_i \quad (i = 0, 1, 2), \\ i_0^*(\beta_{2i}) = \beta_{2i}, \quad i_0^*(2\beta_{2i+1}) = 2\beta_{2i+1}, \end{cases} \qquad \textit{if } t \textit{ is even.}$$

(ii)
$$\begin{cases} i_1^*(\alpha_0) = i_1^*(\alpha_1 + \alpha_2) = 0, \\ i_1^*(\beta_{2i}) = r(\mu^{2i} - 1), \quad i_1^*(2\beta_{2i+1}) = 2r(\mu^{2i+1} - 1), \end{cases} \qquad \textit{if } t \textit{ is odd,}$$

$$\begin{cases} i_1^*(\alpha_i) = 0 \qquad (i=0,1,2), \\ i_1^*(\beta_{2i}) = r(\mu^{2i}-1),\ i_1^*(2\beta_{2i+1}) = 2r(\mu^{2i+1}-1), \end{cases} \qquad \textit{if } t \textit{ is even.}$$

Let $m \geqq 2$, and define $\beta(s)$ in $\widetilde{R}(Q_r)$ $(r=2^{m-1})$ inductively as follows:

$$\beta(0)=\beta_1,\ \beta(s)=\beta(s-1)^2+4\beta(s-1) \quad (s\geqq 1). \tag{2.14}$$

Then, we have the following lemmas.

LEMMA 2.15. $\beta(k+1)-2\sum_{s=1}^{k}\beta(s)\prod_{t=s+1}^{k}(2+\beta(t))=\beta(1)\prod_{t=1}^{k}(2+\beta(t))$ in $\widetilde{R}(Q_r)$.

PROOF. By the induction on k, we can easily verify the equality. q.e.d.

LEMMA 2.16. $P_{m,s}=\beta(s)\prod_{t=s-1}^{m-2}(2+\beta(t))=0$ $(1\leqq s\leqq m)$ *holds in* $\widetilde{R}(Q_r)$.

PROOF. In the similar way to the proof of [9, Lemmas 5.3–4], we have $\beta_{r-1}-\beta_1$ $=\sum_{s=1}^{m-2}(2+\beta_1)\beta(s)\prod_{t=s+1}^{m-2}(2+\beta(t))$ and $(2+\beta_1)\beta(m-1)=2(\beta_{r-1}-\beta_1)$. Hence, by Lemma 2.15, $P_{m,1}=0$ follows. For the case $s\geqq 2$, the equalities

$$P_{m,s}=P_{m,1}\prod_{t=0}^{s-2}(2+\beta(t))$$

and $P_{m,1}=0$ imply $P_{m,s}=0$, q.e.d.

By the definitions of $\beta(s)$, $P_{m,s}$, Lemma 2.16 and Proposition 2.7, we have

LEMMA 2.17. $2P_{m,1}=0$, $\beta_1 P_{m,1}=0$ and $P_{m,s}=0$ $(2\leqq s\leqq m)$ hold in $\widetilde{RO}(Q_r)$.

§3. Some elements in $\widetilde{KO}(S^{4n+3}/Q_t)$

Assume that a topological group G acts freely on a topological space X. Then, the natural projection

$$p: X \longrightarrow X/G$$

define the ring homomorphism (cf. [10, Ch. 12, 5.4])

$$\xi: \widetilde{R}(G) \longrightarrow \widetilde{K}(X/G), \quad \xi: \widetilde{RO}(G) \longrightarrow \widetilde{KO}(X/G) \tag{3.1}$$

Furthermore, if H is the subgroup of G, then the inclusion $i: H \subset G$ and the projections $p': X \longrightarrow X/H$, $i: X/H \longrightarrow X/G$ induce the commutative diagram

(3.2)
$$\begin{tikzcd}
\widetilde{R}(G) \arrow[rrr, "\xi"] \arrow[dr, shift left, "r"] \arrow[ddd, "i^*"'] & & & \widetilde{K}(X/G) \arrow[dl, shift left, "r"] \arrow[ddd, "i^*"] \\
& \widetilde{RO}(G) \arrow[r, "\xi"] \arrow[ul, shift left, "c"] \arrow[d, "i^*"] & \widetilde{KO}(X/G) \arrow[ur, shift left, "c"] \arrow[d, "i^*"] & \\
& \widetilde{RO}(H) \arrow[r, "\xi"'] \arrow[dl, shift left, "c"] & \widetilde{KO}(X/H) \arrow[dr, shift left, "c"] & \\
\widetilde{R}(H) \arrow[rrr, "\xi"'] \arrow[ur, shift left, "r"] & & & \widetilde{K}(X/H) \arrow[ul, shift left, "r"]
\end{tikzcd}$$

$$c\xi=\xi c,\quad r\xi=\xi r,\quad i^*\xi=\xi i^*,\quad ci^*=i^*c,\quad ri^*=i^*r,$$

where c is the complexification and r is the real restriction.

Now, Q_t acts on the unit sphere S^{4n+3} in the quaternion $(n+1)$-space H^{n+1} by the diagonal action

$$q(q_1,\cdots,q_{n+1})=(qq_1,\cdots,qq_{n+1})\quad \text{for } q\in Q_t,\ q_i\in H.$$

Then the natural projection defines the ring homomorphism

$$\xi:\widetilde{RO}(Q_t)\longrightarrow \widetilde{KO}(S^{4n+3}/Q_t)$$

of (3.1), and by using the same letter, we define the elements

(3.3) $\quad \alpha_i=\xi(\alpha_i)\ (i=0,1,2),\ \beta_{2j}=\xi(\beta_{2j})$ and $2\beta_{2j+1}=\xi(2\beta_{2j+1})$

in $\widetilde{KO}(S^{4n+3}/Q_t)$, where α_i, β_{2j+1} and $2\beta_{2j+1}\in\widetilde{RO}(Q_t)$ are the ones in Proposition 2.7.

Consider the orbit manifold S^{4n+3}/G_1 obtained by the restricted action of Q_t to $G_1=Z_q$. As is easily verified, S^{4n+3}/G_1 is homeomorphic to the standard lens space $L^{2n+1}(q)=S^{4n+3}/Z_q$ modulo q. Also, S^{4n+3}/Q_1 is homeomorphic to $L^{2n+1}(4)$.

For $\xi:\widetilde{RO}(Z_k)\longrightarrow\widetilde{KO}(L^{2n+1}(k))$ of (3.1), we have

LEMMA 3.4. *$\xi(r(\mu-1))=r(\eta-1)$, and $\xi(\rho-1)$ is the stable class of the non trivial real line bundle if k is even, where μ and ρ are the elements of Lemmas 2.9, 2.10 and η is the canonical complex line bundle over $L^{2n+1}(k)$.*

PROOF. For $\xi:\widetilde{R}(Z_k)\longrightarrow\widetilde{K}(L^{2n+1}(k))$, we have $\xi(\mu-1)=\eta-1$ by [9, Lemma 3.8]. Thus, the first equality follows from the commutativity $r\xi=\xi r$ in (3.2). Let $k=2l$ and consider the element $c\xi(\rho)$ in $\widetilde{K}(L^{2n+1}(2l))$. Then we see that $c\xi(\rho)=\xi c(\rho)=\xi(\mu^l)=\eta^l$ by (3.2) and the definitions of ρ and μ. Since the first Chern class $c_1(\eta^l)=lc_1(\eta)\neq 0$, $\xi(\rho)$ is the non trivial real line bundle. q.e.d.

REMARK 3.5. *We notice that*

$$\alpha_0=\rho-1 \quad\textit{and}\quad \alpha_1+\alpha_2=r(\mu-1)$$

in $\widetilde{RO}(Q_1)$, and so

$$\alpha_0=\xi(\rho-1)\quad\textit{and}\quad \alpha_1+\alpha_2=r(\eta-1)$$

in $\widetilde{KO}(S^{4n+3}/Q_1)=\widetilde{KO}(L^{2n+1}(4))$.

Let $L_0^{2n+1}(q)$ be the $(4n+2)$-skeleton of $L^{2n+1}(q)$, and $i:L_0^{2n+1}(q)\longrightarrow L^{2n+1}(q)$ be the inclusion. Then we have

LEMMA 3.6. *$i^*\xi(r(\mu-1))=r(\eta-1)$, and $i^*\xi:\widetilde{RO}(Z_q)\longrightarrow\widetilde{KO}(L_0^{2n+1}(q))$ is an epimorphism, where we denote the element $i^*(r(\eta-1))$ in $\widetilde{KO}(L_0^{2n+1}(q))$ by $r(\eta-1)$ for simplicity.*

PROOF. The equality $i^*\xi(r(\mu-1))=r(\eta-1)$ is obtained by Lemma 3.4. Since $\widetilde{KO}(L_0^{2n+1}(q))$ is generated by $r(\eta-1)$ (cf. [11, Prop. 2.11]), $i^*\xi$ is an epimorphism. q.e.d.

Let

(3.7) $$\pi_1 = i^* i_1^* : \widetilde{KO}(S^{4n+3}/Q_t) \longrightarrow \widetilde{KO}(L_0^{2n+1}(q))$$

be the composition of $i_1^* : \widetilde{KO}(S^{4n+3}/Q_t) \longrightarrow \widetilde{KO}(S^{4n+3}/G_1) = \widetilde{KO}(L^{2n+1}(q))$ and $i^* : \widetilde{KO}(L^{2n+1}(q)) \longrightarrow \widetilde{KO}(L_0^{2n+1}(q))$. Then, we have

PROPOSITION 3.8. *π_1 is an epimorphism and*

$$\begin{cases} \pi_1(\alpha_0) = \pi_1(\alpha_1+\alpha_2) = 0, \\ \pi_1(2\beta_1) = 2r(\eta-1), \quad \pi_1(\beta_1^2) = (r(\eta-1))^2, \end{cases} \quad \textit{if } t \textit{ is odd,}$$

$$\begin{cases} \pi_1(\alpha_i) = 0 \quad (i=0,\ 1,\ 2), \\ \pi_1(2\beta_1) = 2r(\eta-1), \quad \pi_1(\beta_1^2) = (r(\eta-1))^2, \end{cases} \quad \textit{if } t \textit{ is even.}$$

PROOF. The equalities except for $\pi_1(\beta_1^2) = (r(\eta-1))^2$ follow from the definition of π_1, (3.2), (3.3), Lemmas 2.13, 3.4 and 3.6. By Propositions 2.5, 2.7, the equality $\beta_1^2 = \beta_2 + \alpha_0 - 4\beta_1$ holds in $\widetilde{RO}(Q_t)$. Since $i_1^*(\beta_2) = r(\eta^2-1)$, $i_1^*(\alpha_0) = 0$ and $i_1^*(2\beta_1) = 2r(\eta-1)$ in $\widetilde{RO}(G_1)$ by Lemma 2.13, there holds the equality $i_1^*(\beta_1^2) = r(\eta^2-1) - 4r(\eta-1)$ in $\widetilde{RO}(G_1)$. On the other hand, $c((r(\eta-1))^2) = (\eta+\eta^{-1}-2)^2 = c(r(\eta^2-1)) - c(4r(\eta-1))$, and the complexification c is monomorphic (cf. (2.1)). Hence

$$r(\eta^2-1) - 4r(\eta-1) = (r(\eta-1))^2 \text{ in } \widetilde{RO}(G_1).$$

Therefore, the desired equality $\pi_1(\beta_1^2) = (r(\eta-1))^2$ follows from (3.2), (3.3), Lemmas 3.4 and 3.6. Also, π_1 is an epimorphism, since $\widetilde{KO}(L_0^{2n+1}(q))$ is an odd torsion group generated by $r(\eta-1) = (1/2)\pi_1(2\beta_1)$ (cf. [11, Prop.2.11]). q.e.d.

For the ring homomorphism

$$\xi : \widetilde{RO}(Q_r) \longrightarrow \widetilde{KO}(S^{4n+3}/Q_r) \quad (r = 2^{m-1} \geqq 2),$$

(3.9) (cf. [17, Th,2.5], [7, Th.1.1 and Cor.1.2]) *ξ is an epimorphism, and*

$$\operatorname{Ker} \xi = \begin{cases} \langle \beta_1^{n+1} RO(Q_r) \rangle & \textit{if } n \textit{ is odd,} \\ \langle \beta_1^{n+1} RSp(Q_r) \rangle & \textit{if } n \textit{ is even,} \end{cases}$$

where $\langle S \rangle$ means the ideal generated by the set S.

By Propositions 2.5–8, we see easily the following

LEMMA 3.10. *Ker ξ in (3.9) is given as follows:*

$$\operatorname{Ker} \xi = \begin{cases} \langle \beta_1^{n+1} \rangle & \textit{if } n \textit{ is odd,} \\ \langle 2\beta_1^{n+1}, \beta_1^{n+2} \rangle & \textit{if } n \textit{ is even.} \end{cases}$$

For the homomorphism

(3.11) $$i_0^* : \widetilde{KO}(S^{4n+3}/Q_t) \longrightarrow \widetilde{KO}(S^{4n+3}/G_0),$$

we have

PROPOSITION 3.12. *i_0^* is an epimorphism and*

$$\begin{cases} i_0^*(\alpha_0) = \alpha_0, \ i_0^*(\alpha_1+\alpha_2) = \alpha_1+\alpha_2, \\ i_0^*(2\beta_1) = 2(\alpha_1+\alpha_2), \ i_0^*(\beta_1^2) = -4\alpha_1^3-10\alpha_1^2-12\alpha_1, \end{cases} \quad \text{if } t \text{ is odd,}$$

$$\begin{cases} i_0^*(\alpha_i) = \alpha_i \ (i=0, 1, 2), \\ i_0^*(2\beta_1) = 2\beta_1, \ i_0^*(\beta_1^2)=\beta_1^2, \end{cases} \quad \text{if } t \text{ is even.}$$

PROOF. By making use of the relations in Proposition 2.5, these equalities follow from Lemma 2.13, (3.2) and (3.3). By Proposition 2.7, Remark 3.5, [12, Th.B] and (3.9), $\widetilde{KO}(S^{4n+3}/G_0)$ is generated by α_0, $\alpha_1+\alpha_2$ if $m=1$, α_0, α_1, $2\beta_1$ and β_1^2 if $m\geqq 2$. Therefore, i_0^* is an epimorphism. q.e.d.

For any integer $n\geqq 0$ and $m\geqq 2$, we define the elements $2\beta(0)$ and $\beta(s)$ $(s\geqq 1)$ in $\widetilde{KO}(S^{4n+3}/Q_r)$ $(r=2^{m-1})$ as follows:

(3.13) $$2\beta(0)=2\beta_1 \text{ and } \beta(s)=\beta(s-1)^2+4\beta(s-1).$$

Then, by (2.14), (3.3), Lemmas 2.15 and 2.16, we have

LEMMA 3.14. (i) $\beta(k+1)-2\sum_{s=1}^{k}\beta(s)\prod_{t=s+1}^{k}(2+\beta(t)) = \beta(1)\prod_{t=1}^{k}(2+\beta(t))$.

(ii) $2P_{m,1}=0$, $\beta_1 P_{m,1}=0$ *and* $P_{m,s}=0$ $(2\leqq s\leqq m)$,

where $P_{m,s}=\beta(s)\prod_{t=s-1}^{m-2}(2+\beta(t))$.

§4. Proof of Theorem 1.4

The cohomology group of the guotient manifold $X=S^{4n+3}/Q_t$ is given as follows:

(4.1) (cf. [3, Ch. XII, §7]) $H^{4i}(X; Z)=Z_{4t}$ if $0<i\leqq n$,

$H^{4i+2}(X; Z)=Z_4$ $(t:\text{odd}), =Z_2\oplus Z_2$ $(t:\text{even})$ if $0\leqq i\leqq n$,

$H^{2i+1}(X; Z)=0$ if $0\leqq i\leqq 2n$, $H^0(X; Z)=H^{4n+3}(X; Z)=Z$,

$H^{4i}(X; Z_2)=H^{4i+3}(X; Z_2)=Z_2$ if $0\leqq i\leqq n$,

$H^{4i+1}(X; Z_2)=H^{4i+2}(X; Z_2)=Z_2 (t:\text{odd}), =Z_2\oplus Z_2$ $(t:\text{even})$ if $0\leqq i\leqq n$.

By (4.1) and the Atiyah–Hirzebruch spectral sequence for $KO(X)$, we have

LEMMA 4.2.

$$\#\widetilde{KO}(S^{4n+3}/Q_t)\leqq \begin{cases} 2^{3n+2-\varepsilon(n)}t^n & \text{if } t \text{ is odd,} \\ 2^{4n+4-2\varepsilon(n)}t^n & \text{if } t \text{ is even,} \end{cases}$$

where $\#A$ denotes the order of a group A and $\varepsilon(n)=0$ if n is even, $=1$ if n is odd.

REMARK 4.3. For the case $t=1$, the additive structure of $\widetilde{KO}(S^{4n+3}/Q_1)=\widetilde{KO}(L^{2n+1}(4))$ is determined in [12, Th.B] and $\#\widetilde{KO}(S^{4n+3}/Q_1)=2^{3n+2-\varepsilon(n)}$ holds.

First, we study the order of $\widetilde{KO}(S^{4n+3}/Q_r)$ $(r=2^{m-1})$.

Let N^k be the k-skeleton of the CW-complex S^{4n+3}/Q_r in [6, Lemma 2.1], and $j: N^k \subset S^{4n+3}/Q_r$ be the inclusion. For an element $a \in \widetilde{KO}(S^{4n+3}/Q_r)$, we denote its image $j^*(a) \in \widetilde{KO}(N^k)$ by the same letter a.

Consider the homomorphism

$$j_l^*: \widetilde{KO}(N^{8k+l}) \longrightarrow \widetilde{KO}(N^{8k+l-1}) \quad (0 \leqq l \leqq 7) \tag{4.4}$$

induced by the inclusion $j_l: N^{8k+l-1} \subset N^{8k+l}$.

Then, we have

LEMMA 4.5 (cf. [7, §4]). *j_0^* is an epimorphism and*

$$\operatorname{Ker} j_0^* = Z_{2^{m+1}}\langle \beta_1^{2k} \rangle.$$

PROOF. By [7, §4], we see that j_0^* is an epimorphism and Ker j_0^* is a cyclic group generated by β_1^{2k}. On the other hand, $2^{m+1}\beta_1^{2k}=0$ in $\widetilde{KO}(N^{8k+3})$ by [17, Prop.5.5]. Thus $2^{m+1}\beta_1^{2k}=0$ in $\widetilde{KO}(N^{8k})$. Consider the homomorphism $j_0^*: \widetilde{K}(N^{8k}) \longrightarrow \widetilde{K}(N^{8k-1})$. Then, Ker $j_0^* = Z_{2^{m+1}}\langle \beta_1^{2k}\rangle$ $(\subset \widetilde{K}(N^{8k}))$ (cf. [6, Lemma 5.4 and Proof of Theorem 1.1]). Therefore, $c(2^m\beta_1^{2k})=2^m\beta_1^{2k} \neq 0$ for the complexification $c: \widetilde{KO}(N^{8k}) \longrightarrow \widetilde{K}(N^{8k})$. These imply that the order of β_1^{2k} is equal to 2^{m+1}. q.e.d.

LEMMA 4.6 (cf. [7, §4]). *j_l^* is isomorphic for $l=7$, 6, 5 and 3.*

LEMMA 4.7 (cf. [7, §4]). *j_4^* is an epimorphism and*

$$\operatorname{Ker} j_4^* = Z_{2^{m+1}}\langle 2\beta_1^{2k+1} \rangle.$$

PROOF. By [7, §4], j_4^* is an epimorphism and Ker j_4^* is a cyclic group generated by $2\beta_1^{2k+1}$. On the other hand, the order of $2\beta_1^{2k+1}$ is equal to 2^{m+1} in $\widetilde{KO}(N^{8k+7})$ by [17, Prop.5.5], and $\widetilde{KO}(N^{8k+7}) \cong \widetilde{KO}(N^{8k+4})$. Thus, we have the desired result. q.e.d.

LEMMA 4.8. *If $a\alpha_1\beta_1^n = x\beta_1^{n+1}$ holds in $\widetilde{R}(Q_r)$ for some $a \in Z$ and $x \in RSp(Q_r)$, then $a\alpha_1\beta_1 = x\beta_1^2$ holds in $\widetilde{R}(Q_r)$.*

PROOF. The statement is trivial for $n=0$. Assume that $n>0$. Since $2^{m+1}(2\beta_1)=0$ in $\widetilde{KO}(S^7/Q_r)$ by Lemmas 4.6 and 4.7, there exists an element $x' \in RO(Q_r)$ such that

$$2^{m+2}\beta_1 = x'\beta_1^2 \quad \text{in} \quad \widetilde{R}(Q_r)$$

by (3.9). Therefore we have

$$a\alpha_1\beta_1^n x'^{n-1} = x\beta_1^{n+1}x'^{n-1}$$

and so $(2^{m+2})^{n-1}a\alpha_1\beta_1 = (2^{m+2})^{n-1}x\beta_1^2$ in $\widetilde{R}(Q_r)$. This implies the desired result, because $\widetilde{R}(Q_r)$ is a free Z-module. q.e.d.

LEMMA 4.9. *If* $a\alpha_1\beta_1 = x\beta_1^2$ *holds in* $\widetilde{R}(Q_r)$ *for some* $a \in Z$ *and* $x \in RSp(Q_r)$, *then* $a \equiv 0 \bmod 4$.

PROOF. By (2.4) and Proposition 2.8, x is uniqely represented as

$$x = 2\varepsilon + 2\varepsilon_0 a_0 + 2\varepsilon_1 a_1 + 2\varepsilon_2 a_2 + \sum_{i=1}^{(r/2)-1} 2\lambda_{2i} b_{2i} + \sum_{i=1}^{r/2} \lambda_{2i-1} b_{2i-1},$$

where ε, ε_0, ε_1, ε_2 and λ_j are some integers. By Proposition 2.3,

$$a\alpha_1\beta_1 = a(2 - 2a_1 - b_1 + b_{r-1}) \text{ and } x\beta_1^2 = x(5 + a_0 - 4b_1 + b_2).$$

Represent $x(5 + a_0 - 4b_1 + b_2)$ by the linear combination of the basis of $R(Q_r)$ by making use of the relations in Propošition 2.5, and compare the constant term and the coefficient of a_0 in $a\alpha_1\beta_1$ with the ones in $x\beta_1^2$. Then, we have

$$2a = 10\varepsilon + 2\varepsilon_0 - 4\lambda_1 + 2\lambda_2 \text{ and } 0 = 10\varepsilon_0 + 2\varepsilon - 4\lambda_1 + 2\lambda_2,$$

and so $a \equiv 0 \bmod 4$. q.e.d.

LEMMA 4.10. *The orders of* $\alpha_0\beta_1^{2k}$ *and* $\alpha_1\beta_1^{2k}$ *are* 4 *in* $\widetilde{KO}(N^{8k+3}) = \widetilde{KO}(S^{8k+3}/Q_r)$.

PROOF. We notice that $\alpha_0\beta_1^{2k} = 2^{2k}\alpha_0$ by Proposition 2.5. Consider the homomorphism $i^* : \widetilde{KO}(S^{8k+3}/Q_r) \longrightarrow \widetilde{KO}(S^{8k+3}/Q_2)$ induced from the inclusion $i : Q_2 \subset Q_r$. Then we have

$$i^*(2\,\alpha_0\beta_1^{2k}) = i^*(2^{2k+1}\alpha_0) = 2^{2k+1}\alpha_0 \neq 0 \text{ in } \widetilde{KO}(S^{8k+3}/Q_2)$$

(cf. [7, Th.1.3]). On the other hand, $4\alpha_0\beta_1^{2k} = 0$ in $\widetilde{KO}(N^{8k+2}) \cong \widetilde{KO}(N^{8k+3})$ by [7, Lemma 4.5]. Thus the order of $\alpha_0\beta_1^{2k}$ is 4 in $\widetilde{KO}(N^{8k+3})$. Also $4\alpha_1\beta_1^{2k} = 0$ in $\widetilde{KO}(N^{8k+2}) \cong \widetilde{KO}(N^{8k+3})$ by [7, Lemma 4.5]. Hence the order of $\alpha_1\beta_1^{2k}$ is 4 in $\widetilde{KO}(N^{8k+3})$ by Lemmas 4.8–9, Proposition 2.7 and (3.9). q.e.d.

LEMMA 4.11 (cf. [7, §4]). j_2^* *is an epimorphism and*

$$\mathrm{Ker}\ j_2^* = Z_2\langle 2\alpha_0\beta_1^{2k}\rangle \oplus Z_2\langle 2\alpha_1\beta_1^{2k}\rangle.$$

PROOF. By [7, §4], j_2^* is an epimorphism. Consider the homomorphism $i^*: \widetilde{KO}(S^{8k+3}/Q_r) \longrightarrow \widetilde{KO}(S^{8k+3}/Q_2)$ induced from the inclusion $i : Q_2 \subset Q_r$. Then

$$i^*(2\alpha_0\beta_1^{2k}) = 2^{2k+1}\alpha_0 \neq 2^{2k+1}\alpha_1 = i^*(2\alpha_1\beta_1^{2k})$$

in $\widetilde{KO}(S^{8k+3}/Q_2)$ by [7, Th.1.3]. Thus $2\alpha_0\beta_1^{2k} \neq 2\alpha_1\beta_1^{2k}$ in $\widetilde{KO}(N^{8k+3}) = \widetilde{KO}(N^{8k+2})$. Hence the desired result for Ker j_2^* follows from Lemma 4.10 and [7, Lemma 4.5]. q.e.d.

LEMMA 4.12 (cf. [7, §4]). j_1^* *is an epimorphism and*

$$\mathrm{Ker}\ j_1^* = Z_2\langle \alpha_0\beta_1^{2k}\rangle \oplus Z_2\langle \alpha_1\beta_1^{2k}\rangle.$$

PROOF. By [7, §4], j_1^* is an epimorphism. Ker j_1^* is givn in [7, Lemma 4.7]. q.e.d.

Summarizing Lemmas 4.5–7, 4.11 and 4.12, we have

PROPOSITION 4.13. (i) $j_l^* : \widetilde{KO}(N^{8k+1}) \longrightarrow \widetilde{KO}(N^{8k+l-1})$ *is an epimorphism and* j_l^*

is an isomorphism for $l=7,\ 6,\ 5$ *and* 3, *and*

$$\operatorname{Ker} j_l^* = \begin{cases} Z_{2^{m+1}}\langle \beta_1^{2k}\rangle & \text{if } l=0, \\ Z_{2^{m+1}}\langle 2\beta_1^{2k+1}\rangle & \text{if } l=4, \\ Z_2\langle \alpha_0\beta_1^{2k}\rangle \oplus Z_2\langle \alpha_1\beta_1^{2k}\rangle & \text{if } l=1, \\ Z_2\langle 2\alpha_0\beta_1^{2k}\rangle \oplus Z_2\langle 2\alpha_1\beta_1^{2k}\rangle & \text{if } l=2. \end{cases}$$

(ii) $\quad \#\widetilde{KO}(S^{4n+3}/Q_r) = 2^{4n+4-2\varepsilon(n)}r^n,$

where $\varepsilon(n)=0$ *if* n *is even,* $=1$ *if* n *is odd.*

Now, we consider the ring homomorphism

$$(4.14) \qquad \pi = i_0^* \oplus \pi_1 : \widetilde{KO}(S^{4n+3}/Q_t) \longrightarrow \widetilde{KO}(S^{4n+3}/Q_r) \oplus \widetilde{KO}(L_0^{2n+1}(q)),$$

where i_0^* and π_1 are the ones of (3.11) and (3.7), respectively.

THEOREM 4.15. (i) *Let* $t=rq$. $r=2^{m-1}$, $m \geqq 1$ *and* q *is odd. Then,* π *in* (4.14) *is a ring isomorphism.*

(ii) $\begin{cases} \pi(\alpha_0)=\alpha_0, \quad \pi(\alpha_1+\alpha_2)=\alpha_1+\alpha_2, \\ \pi(2\beta_1)=2\alpha_1+2\alpha_2+2\bar{\sigma}, \quad \text{if } t \text{ is odd}, \\ \pi(\beta_1^2) = -4\alpha_1^3-10\alpha_1^2-12\alpha_1+\bar{\sigma}^2, \end{cases}$ $\begin{cases} \pi(\alpha_i) = \alpha_i \ (i=0,\ 1,\ 2), \\ \pi(2\beta_1) = 2\beta_1+2\bar{\sigma} \quad \text{if } t \text{ is even}, \\ \pi(\beta_1^2) = \beta_1^2+\bar{\sigma}^2, \end{cases}$

where $\bar{\sigma} = r(\eta-1)$ *is the real restriction of the stable class of the canonical complex line bundle* η *over* $L_0^{2n+1}(q)$ (cf. Lemma 3.6).

PROOF. π_1 and i_0^* are epimorphisms by Propositions 3.8 and 3.12, respectively. On the other hand, by Remark 4.3, Proposition 4.13(ii) and [11, Prop.2.11],

$$\#\widetilde{KO}(S^{4n+3}/Q_r) = \begin{cases} 2^{3n+2-\varepsilon(n)} & \text{if } r=1, \\ 2^{4n+4-2\varepsilon(n)}r^n & \text{if } r \geqq 2, \end{cases} \quad \text{and} \quad \#\widetilde{KO}(L_0^{2n+1}(q)) = q^n.$$

Therefore π in (4.14) is also an epimorphism since q is odd, and so (i) follows from Lemma 4.2.

(ii) follows from the definition of π and Propositions 3.8 and 3.12. q.e.d.

REMARK 4.16. By Proposition 2.7, (3.3), (3.9), [11, Prop.2.11] and Theorem 4.15, *the ring homomorphism*

$$\xi : \widetilde{RO}(Q_t) \longrightarrow \widetilde{KO}(S^{4n+3}/Q_t)$$

is an epimorphism and so the ring $\widetilde{KO}(S^{4n+3}/Q_t)$ *is generated by* α_0, $\alpha_1+\alpha_2$ *if* $t=1$, α_0, $\alpha_1+\alpha_2$, $2\beta_1$ *and* β_1^2 *if* $t \geqq 3$ *is odd,* α_0, α_1, $2\beta_1$ *and* β_1^2 *if* t *is even.* Moreover, by Theorem 4.15(i), Proposition 4.13(ii) and [11, Prop.2.11], we have

$$\# \widetilde{KO}(S^{4n+3}/Q_t) = \begin{cases} 2^{3n+2-\epsilon(n)} t^n & \textit{if } t \textit{ is odd}, \\ 2^{4n+4-2\epsilon(n)} t^n & \textit{if } t \textit{ is even}. \end{cases}$$

Combining Theorem 4.15 and Remark 4.16, we complete the proof of Theorem 1.4.

§5. Some relations in $\widetilde{KO}(S^{4n+3}/Q_r)$ $(r=2^{m-1})$ for odd n

In this section, we give some relations in $\widetilde{KO}(S^{4n+3}/Q_r)$ $(r=2^{m-1}\geqq 2)$ for odd n, which play an important part in the next section.

For the elements $2\beta(0), \beta(s) \in \widetilde{KO}(S^{4n+3}/Q_r)$ in (3.13), we have the following lemmas:

LEMMA 5.1. *For any integers $k_0, \cdots, k_{s-1} \geqq 0$ and $k_s > 0$ $(0 \leqq s \leqq m)$, we have*

$$(1)_s \qquad 2^{m+1-s+h} \prod_{t=0}^{s} \beta(t)^{k_t} = 0 \quad \textit{if } m-s+h \geqq 0,$$

$$(2)_s \qquad 2^{\epsilon(k_0)} \prod_{t=0}^{s} \beta(t)^{k_t} = 0 \quad \textit{if } m-s+h < 0,$$

where $h=h(k_0, \cdots, k_s)=1+[(n-\sum_{t=0}^{s} 2^t k_t)/2^{s-1}]$ and $\epsilon(k_0)=0$ if k_0 is even, $=1$ if k_0 is odd.

PROOF. We prove the lemma by the induction on s and h. Consider the case $s=0$, and suppose that $h(k_0)<0$. Then $k_0 \geqq n+1$ and $\beta_1^{n+1}=0$ by (3.9). Thus $(1)_0$ and $(2)_0$ for $h(k_0)<0$ hold. Suppose that $h=h(k_0)\geqq 0$, and assume that $(1)_0$ and $(2)_0$ hold for any k_0 with $h(k_0)<h$. Since $h=1+2(n-k_0)>0$,

$$2^h \beta(0)^{k_0-1} P_{m,1} = 0$$

by Lemma 3.14, and so

$$(*) \qquad 2^{m+1+h}\beta(0)^{k_0} + 2^{m-1+h}\beta(0)^{k_0+1} + \sum_{I_0} 2^{m-1-j+h}\beta(0)^{k_0-1}\beta(1)\beta(i_1)\cdots\beta(i_j)=0,$$

by (3.13) and the definition of $P_{m,1}$ in Lemma 3.14, where $I_0=\{(i_1, \cdots, i_j): 1\leqq j \leqq m-2, 0\leqq i_1 < \cdots < i_j \leqq m-2\}$. By the inductive hypothesis and (3.13), the second term and the term for any $(i_1, \cdots, i_j) \in I_0$ in $(*)$ vanish. Thus, $(1)_0$ and $(2)_0$ hold.

Suppose that $1\leqq s \leqq m$ and $h=h(k_0, \cdots, k_s)<0$, and assume that $(1)_{s'}$ and $(2)_{s'}$ hold for any s' with $0\leqq s' < s$. In the case $m-s+h\geqq 0$, by (3.13), we have

$$2^{m+1-s+h}\alpha\beta(s)^{k_s} = \sum_{i=0}^{k_s} \binom{k_s}{i} 2^{m+1-s+h+2i}\alpha\beta(s-1)^{2k_s-i},$$

where $\alpha = \prod_{t=0}^{s-1}\beta(t)^{k_t}$. By the assumption,

$$2^{m+1-s+h+2i}\alpha\beta(s-1)^{2k_s-i}=0 \quad (0\leqq i \leqq k_s).$$

This shows that $(1)_s$ holds for $h=h(k_0, \cdots, k_s)<0$ and $m-s+h\geqq 0$. In the case $m-s+h<0$, we can show that $(2)_s$ holds for $h=h(k_0, \cdots, k_s)<0$ in the similar way to the proof of the case $m-s+h\geqq 0$.

Let $1\leqq s \leqq m$ and $h=h(k_0, \cdots, k_s)\geqq 0$, and assume that $(1)_s$ and $(2)_s$ hold for any $k_0, \cdots, k_s$ with $h(k_0, \cdots, k_s)<h$. By Lemma 3.14,

$$2^{h+1}\alpha\beta(s)^{k_s-1}P_{m,s}=0.$$

Hence

$$2^{m+1-s+h}\alpha\beta(s)^{k_s}+2^{m-s+h}\alpha\beta(s-1)\beta(s)^{k_s}+\sum_{I_s}2^{m-s-j+h}(2+\beta(s-1))\alpha\beta(s)^{k_s}\beta(i_1)\cdots\beta(i_j)=0,$$

where $\alpha=\prod_{t=0}^{s-1}\beta(t)^{k_t}$ and $I_s=\{(i_1,\cdots,i_j):1\leqq j\leqq m-1-s,\ s\leqq i_1<\cdots<i_j\leqq m-2\}$. In the similar way to the proof of the case $s=0$, we have $(1)_s$ and $(2)_s$ for $h\geqq 0$ by the the inductive hypothesis. q.e.d.

LEMMA 5.2. *For any integers* $k_0,\cdots,k_{s-1}\geqq 0$ *and* $k_s>l\geqq 0$ $(0\leqq s<m)$,

$$2^{m+1-s+h'}\alpha\beta(s)^{k_s}=(-1)^l 2^{m+1-s+h'+2l}\alpha\beta(s)^{k_s-l}\ \textit{if}\ m-s+h'\geqq 0.$$

Also

$$2^{\varepsilon(k_0)}\alpha\beta(s)^{k_s}=-2^{\varepsilon(k_0)+2}\alpha\beta(s)^{k_s-1}\ \textit{if}\ k_s\geqq 2\ \textit{and}\ m-s+h'<0.$$

Here, $h'=h'(k_0,\cdots,k_s)=[(n-\sum_{t=0}^{s}2^t k_t)/2^s]$ *and* $\alpha=\prod_{t=0}^{s-1}\beta(t)^{k_t}$.

PROOF. We see easily that

$$2^{m+1-s+h'+2l}\alpha\beta(s)^{k_s-l-2}\beta(s+1)=0$$

if $k_s-1\geqq l>0$ and $m-s+h'\geqq 0$, and also

$$2^{\varepsilon(k_0)}\alpha\beta(s)^{k_s-2}\beta(s+1)=0$$

if $k_s\geqq 2$ and $m-s+h'<0$ by Lemma 5.1. Thus, we have the desired results. q.e.d.

LEMMA 5.3. (i) $2^{m+2h}\beta(0)^{k_0}\beta(1)^{k_1}=0$ *if* $m-1+2h\geqq 0$,

$2^{\varepsilon(k_0)}\beta(0)^{k_0}\beta(1)^{k_1}=0$ *if* $m-1+2h<0$.

(ii) $2^{m-s+2+2h}\alpha\beta(s)^{k_s}=0$ *if* $s\geqq 1$ *and* $m-s+1+2h\geqq 0$,

$2^{\varepsilon(k_0)}\alpha\beta(s)^{k_s}=0$ *if* $s\geqq 1$ *and* $m-s+1+2h<0$,

where $h=h(k_0,\cdots,k_s)$ *is the one in Lemma* 5.1 *and* $\alpha=\prod_{t=0}^{s-1}\beta(t)^{k_t}$.

PROOF. (i) By (3.13), we have

$$2^{m+2h}\beta(0)^{k_0}\beta(1)^{k_1}=\sum_{i=0}^{k_1}\binom{k_1}{i}2^{m+2h+2i}\beta(0)^{k_0+2k_1-i},$$

$$2^{\varepsilon(k_0)}\beta(0)^{k_0}\beta(1)^{k_1}=\sum_{i=0}^{k_1}\binom{k_1}{i}2^{\varepsilon(k_0)+2i}\beta(0)^{k_0+2k_1-i}.$$

On the other hand,

$$2^{m+2h+2i}\beta(0)^{k_0+2k_1-i}=0\ (0\leqq i\leqq k_1)\ \text{if}\ m-1+2h\geqq 0,$$

$$2^{\varepsilon(k_0)+2i}\beta(0)^{k_0+2k_1-i}=0\ (0\leqq i\leqq k_1)\ \text{if}\ m-1+2h<0$$

by Lemma 5.1. Thus we have (i).

(ii) is proved in the same manner as the proof of (i) by making use of (3.13) and Lemma 5.1. q.e.d.

LEMMA 5.4. *Suppose that* $m \geqq 3$, $l \geqq 0$ *and* $l \geqq h = h(k_0, k_1)$ *except for* $(l, h) = (0, -1)$. *Then, we have*

$$(1)_h \qquad 2^{m-2+l}\beta(0)^{k_0}\beta(1)^{k_1} + \delta(l)2^{m-1+l}\beta(0)^{k_0+1}\beta(1)^{k_1-1} = 0 \quad \text{if } k_0 \geqq 0,\ k_1 \geqq 2,$$

$$(2)_h \qquad 2^{m-2+l}\beta(0)^{k_0}\beta(1)^{k_1} - \delta(l)2^{m-1+l}\beta(0)^{k_0-1}\beta(1)^{k_1} = 0 \quad \text{if } k_0,\ k_1 \geqq 1,$$

where $\delta(l) = 1$ *if* $l = 1$, $= -1$ *if* $l \neq 1$. *Moreover, we may replace* $\delta(l)$ *in* $(1)_h$ *and* $(2)_h$ *by* ± 1 *if* $l > h$.

PROOF. In the case $h < 0$, each term in $(1)_h$ and $(2)_h$ vanishes by Lemmas 5.1 and 5.3, and so $(1)_h$ and $(2)_h$ hold. We prove the lemma by the induction on $h \geqq 0$. By Lemma 3.14,

$$2^l\beta(0)^{k_0+1}\beta(1)^{k_1-2}P_{m,1} = 0 \quad \text{if } k_0 \geqq 0,\ k_1 \geqq 2,$$

$$2^l\beta(0)^{k_0-1}\beta(1)^{k_1-1}P_{m,1} = 0 \quad \text{if } k_0,\ k_1 \geqq 1.$$

By expanding the left hand sides of the above relations, we have

$$(1) \qquad 2^{m-1+l}\beta(0)^{k_0+1}\beta(1)^{k_1-1} + 2^{m-2+l}\beta(0)^{k_0+2}\beta(1)^{k_1-1} + \textstyle\sum_{I_1} 2^{m-2+l-j}(2+\beta(0))\beta(0)^{k_0+1}\beta(1)^{k_1-1}\beta(i_1)\cdots\beta(i_j) = 0,$$

$$(2) \qquad 2^{m-1+l}\beta(0)^{k_0-1}\beta(1)^{k_1} + 2^{m-2+l}\beta(0)^{k_0}\beta(1)^{k_1} + \textstyle\sum_{I_1} 2^{m-2+l-j}(2+\beta(0))\beta(0)^{k_0-1}\beta(1)^{k_1}\beta(i_1)\cdots\beta(i_j) = 0,$$

where $I_1 = \{(i_1, \cdots, i_j) : 1 \leqq j \leqq m-2,\ 1 \leqq i_1 < \cdots < i_j \leqq m-2\}$. In the case $h = 0$, any term in $\sum_{I_1}$ of (1) and (2) vanishes by Lemma 5.1, and

$$2^{m-2+l}\beta(0)^{k_0+2}\beta(1)^{k_1-1} = 2^{m-2+l}\beta(0)^{k_0}\beta(1)^{k_1} - 2^{m+l}\beta(0)^{k_0+1}\beta(1)^{k_1-1}$$

by (3.13). Thus, we obtain $(1)_0$ and $(2)_0$ from (1) and (2).

Consider the case $h = 1$. Then, by Lemmas 5.1 and 5.3, $\sum_{I_1}$ in (1) is equal to

$$\pm 2^{m-2+l}\beta(0)^{k_0+1}\beta(1)^{k_1} = \pm 2^{m-1+l}\beta(0)^{k_0+2}\beta(1)^{k_1-1} \quad (\text{by } (1)_0).$$

On the other hand

$$2^{m-2+l}\beta(0)^{k_0+2}\beta(1)^{k_1-1} = 2^{m-2+l}\beta(0)^{k_0}\beta(1)^{k_1} - 2^{m+l}\beta(0)^{k_0+1}\beta(1)^{k_1-1} \quad (\text{by } (3.13)).$$

Hence, by (1)

$$2^{m-2+l}\beta(0)^{k_0}\beta(1)^{k_1} - 3\ 2^{m-1+l}\beta(0)^{k_0+1}\beta(1)^{k_1-1} = 0.$$

Since $2^{m+1+l}\beta(0)^{k_0+1}\beta(1)^{k_1-1} = 0$ by Lemma 5.1, we have $(1)_1$. By Lemma 5.1, $\sum_{I_1}$ in (2) is equal to

$$\pm 2^{m-2+l}\beta(0)^{k_0-1}\beta(1)^{k_1+1} = \pm 2^{m-1+l}\beta(0)^{k_0}\beta(1)^{k_1} \quad (\text{by } (3.13) \text{ and Lemma } 5.1).$$

Therefore, we have $(2)_1$.

Suppose $h \geqq 2$. By Lemma 5.1 and $(2)_{h-2}$, any term of $\sum_{I_1}$ in (1) and (2) vanishes, and also

$$2^{m-2+l}\beta(0)^{k_0+2}\beta(1)^{k_1-1} = 2^{m-2+l}\beta(0)^{k_0}\beta(1)^{k_1} - 2^{m+l}\beta(0)^{k_0+1}\beta(1)^{k_1-1} \quad \text{(by (3.13))}.$$

Thus, we have $(1)_h$ and $(2)_h$ for $h \geqq 2$. Since $2^{m+l}\beta(0)^{k_0+1}\beta(1)^{k_1-1} = 0 = 2^{m+l}\beta(0)^{k_0-1}\beta(1)^{k_1}$ if $l > h$ by Lemma 5.1, the last assertion follows. q.e.d.

LEMMA 5.5. *Suppose that* $m \geqq 3$, $l \geqq 2$ *and* $l \geqq h = h(k_0, k_1)$ *except for* $(l, h) = (2, 1)$. *Then*

$$\left.\begin{array}{ll} (3)_{h=3} & 3\, 2^m\beta(0)^{k_0}\beta(1)^{k_1} + 2^{m+1}\beta(0)^{k_0+1}\beta(1)^{k_1-1} = 0, \\ (3)_{h\neq 3} & 2^{m-3+l}\beta(0)^{k_0}\beta(1)^{k_1} - 2^{m-2+l}\beta(0)^{k_0+1}\beta(1)^{k_1-1} = 0, \end{array}\right\} \quad \text{if } k_0 \geqq 0,\ k_1 \geqq 2,$$

$$\left.\begin{array}{ll} (4)_{h=3} & 3\, 2^m\beta(0)^{k_0}\beta(1)^{k_1} - 2^{m+1}\beta(0)^{k_0-1}\beta(1)^{k_1} = 0, \\ (4)_{h\neq 3} & 2^{m-3+l}\beta(0)^{k_0}\beta(1)^{k_1} + 2^{m-2+l}\beta(0)^{k_0-1}\beta(1)^{k_1} = 0, \end{array}\right\} \quad \text{if } k_0,\ k_1 \geqq 1.$$

PROOF. By Lemma 3.14, we have

$$2^{l-1}\beta(0)^{k_0+1}\beta(1)^{k_1-2}P_{m,1} = 0 \qquad \text{if } k_0 \geqq 0,\ k_1 \geqq 2.$$

By expanding the left hand side of the above relation,

$$\begin{aligned}(3)\qquad & 2^{m-2+l}\beta(0)^{k_0+1}\beta(1)^{k_1-1} + 2^{m-3+l}\beta(0)^{k_0+2}\beta(1)^{k_1-1} \\ & + \textstyle\sum_{I_1} 2^{m-3-j+l}(2+\beta(0))\beta(0)^{k_0+1}\beta(1)^{k_1-1}\beta(i_1)\cdots\beta(i_j) = 0,\end{aligned}$$

where $I_1 = \{(i_1, \cdots, i_j) : 1 \leqq j \leqq m-2,\ 1 \leqq i_1 < \cdots < i_j \leqq m-2\}$. In $\sum_{I_1}$ of (3), the terms for $j \geqq 3$ vanish by Lemma 5.1, and also the terms for $j=2$ vanish by Lemmas 5.1, 5.3 and $(2)_{h-4}$ in Lemma 5.4. Thus, $\sum_{I_1}$ in (3) is equal to

$$2^{m-4+l}(2+\beta(0))\beta(0)^{k_0+1}\beta(1)^{k_1} = \begin{cases} 0 & \text{if } l \geqq h \neq 3, \\ \pm 2^{m+1}\beta(0)^{k_0+1}\beta(1)^{k_1} = \pm 2^{m+2}\beta(0)^{k_0+2}\beta(1)^{k_1-1} & \text{if } l = h = 3, \end{cases}$$

by Lemmas 5.1, 5.3 and 5.4. On the other hand, by (3.13)

$$2^m\beta(0)^{k_0+2}\beta(1)^{k_1-1} = 2^m\beta(0)^{k_0}\beta(1)^{k_1} - 2^{m+2}\beta(0)^{k_0+1}\beta(1)^{k_1-1},$$

and by Lemma 5.1,

$$2^{m+4}\beta(0)^{k_0+1}\beta(1)^{k_1-1} = 0 \ \text{ if } h = 3.$$

Therefore, we have $(3)_h$.

Also, we have

$$2^{l-1}\beta(0)^{k_0-1}\beta(1)^{k_1-1}P_{m,1} = 0 \ \text{ if } k_0,\ k_1 \geqq 1,$$

by Lemma 3.14, and so

$$(4)\ 2^{m-2+l}\beta(0)^{k_0-1}\beta(1)^{k_1} + 2^{m-3+l}\beta(0)^{k_0}\beta(1)^{k_1} + \textstyle\sum_{I_1} 2^{m-3-j+l}(2+\beta(0))\beta(0)^{k_0-1}\beta(1)^{k_1}\beta(i_1)\cdots\beta(i_j) = 0.$$

In the similar way to the proof of $(3)_h$, the terms for $j \geqq 2$ in $\sum_{I_1}$ of (4) vanish, and $\sum_{I_1}$ of (4) is equal to

$$\begin{cases} 0 & \text{if } l \geqq h \neq 3, \\ \pm 2^{m+2}\beta(0)^{k_0}\beta(1)^{k_1} & \text{if } l = h = 3. \end{cases}$$

Hence, we have $(4)_h$. q.e.d.

LEMMA 5.6. *Let k_0 and k_1 be non negative integers. Then*

$$2^{2-\varepsilon(k_0)}\beta(0)^{k_0}\beta(1)^{k_1+1} + 2^{1-\varepsilon(k_0)}\beta(0)^{k_0+1}\beta(1)^{k_1+1} = 0$$

in $\widetilde{KO}(S^{4n+3}/Q_2)$, where $\varepsilon(k_0)=0$ if k_0 is even, $=1$ if k_0 is odd.

PROOF. By Lemma 3.14,

$$2^{1-\varepsilon(k_0)}\beta(0)^{k_0}\beta(1)^{k_1}P_{2,1} = 0,$$

and $P_{2,1} = (2+\beta(0))\beta(1)$ by the definition of $P_{2,1}$. Therefore, the desired result follows. q.e.d.

LEMMA 5.7. *Let $2 \leqq s \leqq m-2$, $l \geqq -1$ and $l \geqq h = h(k_0, \cdots, k_s)$. Then*

$(5)_{h\leqq -1}$ $\quad 2^{m-s-1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s} \pm 2^{m-s+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1} = 0,$

$(5)_{h\geqq 0}$ $\quad 2^{m-s-1+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s} - \bar{\delta}(l)2^{m-s+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1} = 0,$

if $k_0, \cdots, k_{s-1} \geqq 0$ and $k_s \geqq 2$,

$(6)_{h\leqq -1}$ $\quad 2^{m-s-1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s} \pm 2^{m-s+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}-1}\beta(s)^{k_s} = 0,$

$(6)_{h\geqq 0}$ $\quad 2^{m-s-1+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s} + \bar{\delta}(l)2^{m-s+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}-1}\beta(s)^{k_s} = 0,$

if $k_0, \cdots, k_{s-2} \geqq 0$ and $k_{s-1}, k_s \geqq 1$,

where $\alpha = \prod_{t=0}^{s-2}\beta(t)^{k_t}$ and $\bar{\delta}(l) = -1$ if $l=0$, $=1$ if $l \geqq 1$. Moreover, we may replace $\bar{\delta}(l)$ by ± 1 if $l > h$ or k_0 is an odd integer.

PROOF. First we consider the case $h \leqq -1$. By Lemma 3.14,

$$2^{\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-2}P_{m,s} = 0 \quad \text{if } k_{s-1} \geqq 0,\ k_s \geqq 2.$$

Thus, we have

$$(5) \quad 2^{m-s+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1} + 2^{m-s-1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+2}\beta(s)^{k_s-1}$$
$$+ \textstyle\sum_{I_s} 2^{m-s-1-j+\varepsilon(k_0)}(2+\beta(s-1))\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1}\beta(i_1)\cdots\beta(i_j) = 0,$$

where $I_s = \{(i_1, \cdots, i_j) : 1 \leqq j \leqq m-1-s,\ s \leqq i_1 < \cdots < i_j \leqq m-2\}$. $\sum_{I_s}$ of (5) vanishes by Lemma 5.1, and

$$2^{m-s-1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+2}\beta(s)^{k_s-1} = \pm 2^{m-s-1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s}$$

by (3.13) and Lemma 5.1. This implies $(5)_{h\leqq -1}$. In the similar way to the proof of $(5)_{h\leqq -1}$, $(6)_{h\leqq -1}$ is obtained from the relation

$$(6) \quad 2^{m-s+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}-1}\beta(s)^{k_s} + 2^{m-s-1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s}$$

$$+\sum_{I_s} 2^{m-s-1-j+\varepsilon(k_0)}(2+\beta(s-1))\alpha\beta(s-1)^{k_{s-1}-1}\beta(s)^{k_s}\beta(i_1)\cdots\beta(i_j)=0$$

which is the expansion of the relation

$$2^{\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}-1}\beta(s)^{k_s-1}P_{m,s}=0$$

in Lemma 3.14.

In the case $h=0$, the terms for $(i_1, \cdots, i_j)\in I_s$ in $\sum_{I_s}$ of (5) vanish except for (s) by Lemma 5.1 and so $\sum_{I_s}$ of (5) is equal to

$$\pm 2^{m-s-1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s} \quad \text{(by Lemma 5.3)}$$
$$=\pm 2^{m-s+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+2}\beta(s)^{k_s-1} \quad \text{(by } (5)_{-1}).$$

On the other hand,

$$2^{m-s-1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+2}\beta(s)^{k_s-1}$$
$$=2^{m-s-1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s}\pm 2^{m-s+1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1}$$

by (3.13) and Lemma 5.1. These imply $(5)_0$ from (5). $(6)_0$ is obtained from (6) in the similar way to the proof of $(5)_0$.

Suppose $h\geqq 1$ and consider the relation $2^l\times(5)$

$$2^{m-s+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1}+2^{m-s-1+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+2}\beta(s)^{k_s-1}$$
$$+\sum_{I_s} 2^{m-s-1+l+\varepsilon(k_0)-j}(2+\beta(s-1))\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1}\beta(i_1)\cdots\beta(i_j)=0.$$

The terms for $(i_1, \cdots, i_j)\in I_s$ vanish by Lemma 5.1 except for (s), and also the term for (s) vanishes by $(6)_{h-2}$. Therefore, we have $(5)_{h\geq 1}$. $(6)_{h\geq 1}$ follows from the relation $2^l\times$ (6) in the similar way to the proof of $(5)_{h\geq 1}$. q.e.d.

LEMMA 5.8. *Let* $m\geqq 3$, $k_{m-2}\geqq 0$ *and* $k_{m-1}\geqq 0$. *Then*

$$2^{\varepsilon(k_0)}\alpha\beta(m-2)^{k_{m-2}+1}\beta(m-1)^{k_{m-1}+1}+2^{\varepsilon(k_0)+1}\alpha\beta(m-2)^{k_{m-2}}\beta(m-1)^{k_{m-1}+1}=0,$$

where α *is any monomial of* $\beta(0), \cdots, \beta(m-3)$.

PROOF. The result follows immediately from the relation

$$2^{\varepsilon(k_0)}\alpha\beta(m-2)^{k_{m-2}}\beta(m-1)^{k_{m-1}}P_{m,m-1}=0$$

and the definition of $P_{m,m-1}$ in Lemma 3.14. q.e.d.

LEMMA 5.9. *Let* $2\leqq s\leqq m-2$, $l\geqq 2$ *and* $l\geqq h=h(k_0, \cdots, k_s)$. *Then the following relations hold*:

$(7)_{h(l=2)}$ $\quad 2^{m-s+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s}+3\, 2^{m-s+1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1}=0,$

$(7)_{h(l\geq 3)}$ $\quad 2^{m-s-2+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s}-2^{m-s-1+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta^{k_s-1}=0,$

if $k_0, \cdots, k_{s-1}\geqq 0$, $k_s\geqq 2$,

$(8)_{h(l=2)}$ $\quad 2^{m-s+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s}-3\, 2^{m-s+1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}-1}\beta(s)^{k_s}=0,$

$(8)_{h(l\geqq 3)}$ $\quad 2^{m-s-2+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s}+2^{m-s-1+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}-1}\beta(s)^{k_s}=0,$

if $k_0, \cdots, k_{s-2}\geqq 0$ *and* $k_{s-1},\ k_s\geqq 1$,

where $\alpha=\prod_{t=0}^{s-2}\beta(t)^{k_t}$.

PROOF. By Lemma 3.14, we have

$$2^{l-1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-2}P_{m,s}=0.$$

Therefore

$$2^{m-s-1+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1}+2^{m-s-2+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+2}\beta(s)^{k_s-1}$$
$$+\textstyle\sum_{I_s}2^{m-s-2+l+\varepsilon(k_0)-j}(2+\beta(s-1))\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1}\beta(i_1)\cdots\beta(i_j)=0.$$

If k_0 is odd, any term in $\sum_{I_s}$ vanishes by Lemmas 5.1, 5.3 and 5.7. Also, by (3.13)

$$2^{m-s-2+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+2}\beta(s)^{k_s-1}=2^{m-s-2+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s}$$
$$-2^{m-s+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1}$$

and

$$2^{m-s+4}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1}=0 \quad \text{if } l=2$$

by Lemma 5.1. Thus, we have $(7)_h$ in the case k_0 is odd. In the case k_0 is even, the terms for $(i_1,\cdots,i_j)\in\sum_{I_s}$ except for (s) vanish by Lemmas 5.1, 5.3, and 5.7. Thus, $\sum_{I_s}$ is equal to

$$2^{m-s-3+l}(2+\beta(s-1))\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s}=\begin{cases}0 & \text{if } l\geqq 3,\\ \pm 2^{m-s+1}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s} & \text{if } l=2,\end{cases}$$

by Lemmas 5.7 and 5.1. Also, by Lemma 5.7

$$\pm 2^{m-s+1}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s}=\pm 2^{m-s+2}\alpha\beta(s-1)^{k_{s-1}+2}\beta(s)^{k_s-1} \quad \text{if } l=2.$$

Therefore, we have $(7)_h$ in the case k_0 is even. $(8)_h$ follows from the relation

$$2^{l-1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}-1}\beta(s)^{k_s-1}P_{m,s}=0$$

given by Lemma 3.14 in the similar way to the proof of $(7)_h$ above. q.e.d.

LEMMA 5.10. *Suppose* $m\geqq 3$, $l\geqq 0$ *and* $l\geqq h=h(k_0, k_1)$. *Then, the following relations hold for any* $k_0\geqq 0$ *and* $k_1\geqq 2$:

$$2^{m-2+l}\beta(0)^{k_0}\beta(1)^{k_1}=2^{m+l}\beta(0)^{k_0}\beta(1)^{k_1-1} \quad \textit{if } l=0,\ 1 \textit{ and } (l,h)\neq(0,-1)$$
$$2^{m-2+l}\beta(0)^{k_0}\beta(1)^{k_1}=-2^{m+l}\beta(0)^{k_0}\beta(1)^{k_1-1} \quad \textit{if } l\geqq 2,$$
$$2^{m-3+l}\beta(0)^{k_0}\beta(1)^{k_1}=3\ 2^{m-1+l}\beta(0)^{k_0}\beta(1)^{k_1-1} \quad \textit{if } l=2,\ 3 \textit{ and } (l,h)\neq(2,1),$$
$$2^{m-3+l}\beta(0)^{k_0}\beta(1)^{k_1}=-2^{m-1+l}\beta(0)^{k_0}\beta(1)^{k_1-1} \quad \textit{if } l\geqq 4.$$

PROOF. These relations follow immediately from Lemmas 5.4 and 5.5. q.e.d.

LEMMA 5.11. *Suppose* $2 \leqq s \leqq m-2$, $l \geqq -1$ *and* $l \geqq h(k_0, \cdots, k_s)$. *Then the following relations hold for any* $k_0, \cdots, k_{s-1} \geqq 0$ *and* $k_s \geqq 2$:

$$2^{m-s-1+\varepsilon(k_0)}\alpha\beta(s)^{k_s} = \pm 2^{m-s+1+\varepsilon(k_0)}\alpha\beta(s)^{k_s-1} \quad \text{if } l=-1,$$

$$2^{m-s-1+\varepsilon(k_0)}\alpha\beta(s)^{k_s} = 2^{m-s+1+\varepsilon(k_0)}\alpha\beta(s)^{k_s-1} \quad \text{if } l=0,$$

$$2^{m-s-1+l+\varepsilon(k_0)}\alpha\beta(s)^{k_s} = -2^{m-s+1+l+\varepsilon(k_0)}\alpha\beta(s)^{k_s-1} \quad \text{if } l\geqq 1,$$

$$2^{m-s+\varepsilon(k_0)}\alpha\beta(s)^{k_s} = 3\ 2^{m-s+2+\varepsilon(k_0)}\alpha\beta(s)^{k_s-1} \quad \text{if } l=2,$$

$$2^{m-s-2+l+\varepsilon(k_0)}\alpha\beta(s)^{k_s} = -2^{m-s+l+\varepsilon(k_0)}\alpha\beta(s)^{k_s-1} \quad \text{if } l\geqq 3,$$

where $\alpha = \prod_{t=0}^{s-1}\beta(t)^{k_t}$.

PROOF. We see easily the desired results by Lemmas 5.7 and 5.9.

q.e.d.

LEMMA 5.12. *Suppose* $l \geqq 0$, $l \geqq h = h(k_0, \cdots, k_{s-1}, 1)$, $k_0, \cdots, k_{s-2} \geqq 0$ *and* $k_{s-1} \geqq 1$. *Then we have the following relations*:

(i) $$2^{m+l}\beta_1^{k_0+1} + (1 \pm 2^{l+1})2^{m+2+l}\beta_1^{k_0} = 0 \quad \text{if } s=1 \text{ and } m \geqq 2.$$

Moreover

$$2^{m-1}\beta_1^{n} + 3\ 2^{m+1}\beta_1^{n-1} = 0 \quad \text{if } s=1,\ m\geqq 2 \text{ and } h=0,$$

$$2^2\beta_1^{n-1} + 5\ 2^4\beta_1^{n-2} = 0 \quad \text{if } s=1,\ m=2 \text{ and } h=1,$$

$$2^m\beta_1^{n-1} - 3\ 2^{m+2}\beta_1^{n-2} = 0 \quad \text{if } s=1,\ m\geqq 3 \text{ and } h=1.$$

(ii) $$2^{m-1+l}\alpha\beta(1)^{k_1+1} + (1 \pm 2^{l+2-\varepsilon(k_0)})2^{m+1+l}\alpha\beta(1)^{k_1} = 0 \quad \text{if } s=2 \text{ and } m\geqq 3.$$

(iii) $$2^{m-s+1+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1} + (1 \pm 2^{l+2})2^{m-s+3+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}} = 0 \quad \text{if } 3 \leqq s \leqq m-1.$$

Here, $\alpha = \prod_{t=0}^{s-2}\beta(t)^{k_t}$ *in* (ii) *and* (iii).

PROOF. (i) The first relation holds obviously by Lemma 5.1 if $h = n-1-k_0 < 0$. Consider the case $m=2$. By Lemma 5.6 and (3.13), we have

(5.13) $$2^{\varepsilon(k_0)}\beta_1^{k_0+2} + 3\ 2^{\varepsilon(k_0)+1}\beta_1^{k_0+1} + 2^{\varepsilon(k_0)+3}\beta_1^{k_0} = 0 \quad \text{if } k_0 \geqq 1.$$

When $h = n-1-k_0 = 0$, the second relation

$$2\beta_1^n + 3\ 2^3\beta_1^{n-1} = 0$$

follows from (5.13) and Lemma 5.1. Also, the first relation for $h=0$ is obtained from the second one by Lemma 5.1. When $h = n-1-k_0 = 1$, the third relation follows from (5.13), Lemma 5.1 and the second one. The first relation for $h=1$ is shown from the third one by Lemma 5.1.

Now, consider the case $m \geqq 3$. In the relation $(2)_h$ of Lemma 5.4, put $k_1 = 1$. Then, we have the second relation and also the first one for $h=0$ by Lemma 5.1. The forth relation follows from the first one for $h=0$ and Lemma 5.1. The first relation for $h=1$ is the immediate consequence of the forth one.

Suppose $m \geqq 2$ and $h \geqq 2$. We shall prove the first relation for $h \geqq 2$ by the induction on h. By (5.13) if $m=2$ and $(2)_h$ of Lemma 5.4 if $m \geqq 3$, we have

$$2^{m-1+l}\beta_1^{k_0+2}+3\ 2^{m+l}\beta_1^{k_0+1}+2^{m+2+l}\beta_1^{k_0}=0.$$

By the inductive assumption,

$$2^{m-1+l}(4+\beta_1)\beta_1^{k_0+1}=\pm 2^{m+1+2l}\beta_1^{k_0+1}.$$

Therefore, we have

$$(1\pm 2^{l+1})2^{m+l}\beta_1^{k_0+1}+2^{m+2+l}\beta_1^{k_0}=0,$$

and so

$$2^{m+l}\beta_1^{k_0+1}+(1\pm 2^{l+1})2^{m+2+l}\beta_1^{k_0}=0$$

by Lemma 5.1. Thus, we complete the proof of (i).

(ii), (iii) In the case $h<0$, (ii) and (iii) are obtained from Lemmas 5.10, 5.11 and 5.1. Consider the case $2 \leqq s \leqq m-2$ and $h \geqq 0$. We shall prove (ii), (iii) for $h \geqq 0$ by the induction on h. Let $h=0$ and put $k_s=1$ in the relation $(6)_0$ of Lemma 5.7. Then, we have

$$(5.14)\qquad 2^{m-s-1+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+2}+2^{m-s+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}-2^{m-s+2+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}=0.$$

By Lemma 5.3, $2^{m-2}\alpha\beta(1)^{k_1+2}=0$ if $h=h(k_0, k_1, 1)=0$ and k_0 is odd. Thus, (ii) for $h=0$ and odd k_0 is obtained from (5.14) with $s=2$ and Lemma 5.1. (ii) for $h=0$ and even k_0 follows from $2\times$(5.14) with $s=2$ and Lemma 5.1, since

$$2^{m-2}\alpha\beta(1)^{k_1+2}=2^{m}\alpha\beta(1)^{k_1+1}\quad \text{if } h=h(k_0, k_1, 1)=0$$

by Lemma 5.10. Moreover, (iii) for $h=0$ and $3 \leqq s \leqq m-2$ follows from $2\times$(5.14), Lemmas 5.1 and 5.11. Let $h \geqq 1$ and put $k_s=1$ in the relation $(6)_h$ of Lemma 5.7. Then, we have

$$(5.15)\qquad 2^{m-s-1+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+2}+3\ 2^{m-s+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}+2^{m-s+2+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}=0$$

(ii) for $h \geqq 1$ and (iii) for $h \geqq 1$ and $3 \leqq s \leqq m-2$ follow from (5.15) and Lemma 5.1 by the induction on h. Consider the case $s=m-1$ and $h \geqq 0$. By Lemma 5.8 and (3.13), we have

$$(5.16)\qquad 2^{\varepsilon(k_0)}\alpha\beta(m-2)^{k_{m-2}+2}+3\ 2^{\varepsilon(k_0)+1}\alpha\beta(m-2)^{k_{m-2}+1}+2^{\varepsilon(k_0)+3}\alpha\beta(m-2)^{k_{m-2}}=0.$$

(iii) for $h \geqq 0$ and $s=m-1$ can be proved inductively by making use of (5.16), Lemmas 5.1, 5.3, 5.10 and 5.11 in the similar way to the proof of (iii) for $h \geqq 1$ and $3 \leqq s \leqq m-2$. q.e.d.

§6. Basic relations concerned with an additive base of $\widetilde{KO}(S^{4n+3}/Q_r)$ ($r=2^{m-1}$) for odd n

In this section, we prove some basic relations concerned with an additive base of $\widetilde{KO}(S^{4n+3}/Q_r)$ ($r=2^{m-1}$) for odd n by making use of the relations given in §5.

Let s, k and d be the integers which satisfy

$$0 \leqq s \leqq m-2,\ 2^s(k-1) \leqq n-d < 2^s k,\ k \geqq 2 \text{ and } d \geqq 0. \tag{6.1}$$

Then we have the following lemmas.

LEMMA 6.2. *Suppose* $1 \leqq s \leqq m-2$, $k=2k' \geqq 2$ *and* d *is even under the assumption* (6.1). *Then*

$$2^{m-s-2}\beta_1^d\beta(s)^k = \textstyle\sum_{t=1}^{s} 2^{m-s-4+2^{t+1}k}\beta_1^d\beta(s-t).$$

PROOF. Let $u=s-t$ $(1 \leqq t \leqq s)$. Then, by (3.13), we have

$$2^{m-s-2}\beta_1^d(\beta(u+1)^{2^{t-1}k}-\beta(u)^{2^t k}) = \textstyle\sum_{i=1}^{2^{t-1}k}\binom{2^{t-1}k}{i}2^{m-s-2+2i}\beta_1^d\beta(u)^{2^t k-i}.$$

The i-th term except for $i=1, 2$ is equal to

$$(-1)^{i-1}\binom{2^{t-1}k}{i}2^{m-s-4+2^{t+1}k}\beta_1^d\beta(u)$$

by Lemma 5.2. The i-th term form $i=1, 2$ is equal to

$$\begin{aligned}
&\binom{2^{t-1}k}{i}2^{m-s-2+2i}\beta_1^d\beta(u)^{2-i}(\beta(u+1)-2^2\beta(u))^{2^{t-1}k-1} \qquad \text{(by (3.13))}\\
&=\binom{2^{t-1}k}{i}2^{m-s-2+2i}\beta_1^d\beta(u)^{2-i}\beta(u+1)^{2^{t-1}k-1}\\
&\qquad +\textstyle\sum_{j=1}^{2^{t-1}k-1}(-1)^j\binom{2^{t-1}k}{i}\binom{2^{t-1}k-1}{j}2^{m-s-2+2i+2j}\beta_1^d\beta(u)^{2-i+j}\beta(u+1)^{2^{t-1}k-1-j}\\
&=\binom{2^{t-1}k}{i}2^{m-s-2+2i}\beta_1^d\beta(u)^{2-i}\beta(u+1)^{2^{t-1}k-1}\\
&\qquad +(-1)^{2^{t-1}k-1}\binom{2^{t-1}k}{i}2^{m-s-4+2i+2^t k}\beta_1^d\beta(u)^{2^{t-1}k+1-i} \qquad \text{(by Lemma 5.1)}\\
&=\pm\binom{2^{t-1}k}{i}2^{m-s-6+2i+2^t k}\beta_1^d\beta(u)^{2-i}\beta(u+1)\\
&\qquad +(-1)^{i-1}\binom{2^{t-1}k}{i}2^{m-s-4+2^{t+1}k}\beta_1^d\beta(u) \qquad \text{(by Lemmas 5.2 and 5.1).}
\end{aligned}$$

By Lemma 5.1

$$\binom{2^{t-1}k}{i}2^{m-s-6+2i+2^t k}\beta_1^d\beta(u)^{2-i}\beta(u+1) = \begin{cases} 0 & \text{if } i=1 \text{ or } 2 \text{ and } k'\text{: even} \geqq 2,\\ 2^{m-s+t-4+2^t k}\beta_1^d\beta(u)\beta(u+1) & \text{if } i=1 \text{ and } k'\text{: odd} \geqq 1,\\ 2^{m-s+t-3+2^t k}\beta_1^d\beta(u+1) & \text{if } i=2 \text{ and } k'\text{: odd} \geqq 1.\end{cases}$$

On the other hand

$$2^{m-(u+1)-3+2^{t}k}\beta_1^d(2+\beta(u))\beta(u+1)=0 \quad \text{(by Lemmas 5.4 and 5.7).}$$

Therefore, we have

$$2^{m-s-2}\beta_1^d(\beta(u+1)^{2^{t-1}k}-\beta(u)^{2^t k})=2^{m-s-4+2^{t+1}k}\beta_1^d\beta(u) \quad (0\leqq u\leqq s-1).$$

Summarizing these terms for $0\leqq u\leqq s-1$, we have the desired result, since $2^{m-s-2}\beta_1^{d+2^s k}=0$ by Lemma 5.1. q.e.d.

LEMMA 6.3. *Suppose* $1\leqq s\leqq m-3$, $k=2k'$ *and* d *is even under the assumption* (6.1). *Then*

$$2^{m-s-2}\beta_1^d\beta(s)^k=2^{m-s-4+k}\beta_1^d\beta(s+1)-2^{m-s-4+2k}\beta_1^d\beta(s).$$

PROOF. The result for $k'=1$ follows immediately from (3.13). Suppose $k'\geqq 2$. Then, by (3.13), we have

$$(*) \qquad 2^{m-s-2}\beta_1^d(\beta(s+1)^{k'}-\beta(s)^k)=\textstyle\sum_{i=1}^{k'}\binom{k'}{i}2^{m-s-2+2i}\beta_1^d\beta(s)^{k-i}.$$

By Lemmas 5.10, 5.11 and 5.2,

$$2^{m-s-2+2i}\beta_1^d\beta(s)^{k-i}=(-1)^{i-1}2^{m-s-4+2k}\beta_1^d\beta(s)$$

for $2\leqq i\leqq k'$. The first term in the right hand side of $(*)$ is equal to

$$k'2^{m-s}\beta_1^d\beta(s)^{k-1}=-3k'2^{m-s-4+2k}\beta_1^d\beta(s)$$

by Lemmas 5.10 and 5.11. Therefore, the right right hand side of $(*)$ is equal to

$$2^{m-s-4+2k}\beta_1^d\beta(s)-k'2^{m-s-2+2k}\beta_1^d\beta(s).$$

On the other hand

$$2^{m-s-2}\beta_1^d\beta(s+1)^{k'}=(-1)^{k'}2^{m-s-4+k}\beta_1^d\beta(s+1)$$

by Lemma 5.11. Hence, by Lemma 5.1, the desired relation for even k' holds, and also the relation

$$(**) \qquad 2^{m-s-2}\beta_1^d\beta(s)^k=-2^{m-s-4+k}\beta_1^d\beta(s+1)+3\,2^{m-s-4+2k}\beta_1^d\beta(s)$$

holds if k' is odd. Moreover, by (3.13) and Lemma 5.12

$$2^{m-s-3+k}\beta_1^d\beta(s+1)=2^{m-s-1+k}\beta_1^d\beta(s)+2^{m-s-3+k}\beta_1^d\beta(s)^2=\pm 2^{m-s-2+2k}\beta_1^d\beta(s).$$

Thus, the desired result for odd k' follows from $(**)$. q.e.d.

LEMMA 6.4. *Suppose* $s=m-2\geqq 1$, $k=2k'$ *and* d *is even under the assumption* (6.1). *Then*

$$\beta_1^d\beta(m-2)^k=\begin{cases}\beta_1^d\beta(m-1)-2^2\beta_1^d\beta(m-2) & \text{if } k'=1,\\[2ex] -2^{k-2}\beta_1^d\beta(m-1)-2^{2k-2}\beta_1^d\beta(m-2) & \text{if } k'\geqq 2.\end{cases}$$

PROOF. The result for $k'=1$ follows immediately from (3.13).

Suppose $k' \geqq 2$. Then, in the same manner as the proof of Lemma 6.3, we have

$$\beta_1^d(\beta(m-1)^{k'}-\beta(m-2)^{k'}) = 2^{2k-2}\beta_1^d\beta(m-2)-k'2^{2k}\beta_1^d\beta(m-2).$$

Since $P_{m,m}=\beta(m)=\beta(m-1)^2+2^2\beta(m-1)=0$ by (3.13) and Lemma 3.14, we have

$$\beta_1^d\beta(m-1)^{k'} = (-1)^{k'-1}2^{k-2}\beta_1^d\beta(m-1).$$

Therefore, we see that

$$(*) \qquad \beta_1^d\beta(m-2)^{k'} = (-1)^{k'-1}2^{k-2}\beta_1^d\beta(m-1)-2^{2k-2}\beta_1^d\beta(m-2)+k'2^{2k}\beta_1^d\beta(m-2).$$

In the case k' is even, the last term in $(*)$ vanishes by Lemma 5.1, and so the desired relation holds. Suppose k' is odd. Then the last term of $(*)$ is equal to

$$\pm 2^{2k}\beta_1^d\beta(m-2).$$

by Lemma 5.1. On the other hand

$$2^{k-1}\beta_1^d\beta(m-1) = 2^{k-1}\beta_1^d\beta(m-2)^2+2^{k+1}\beta_1^d\beta(m-2) = \pm 2^{2k}\beta_1^d\beta(m-2)$$

by (3.13) and Lemma 5.12. Thus, the desired relation for odd k' follows from $(*)$.

q.e.d.

LEMMA 6.5. *Suppose $s=0$, $k=2k'$ and d is even under the assumption* (6.1). *Then, we have*

$$\beta_1^d\beta(1)-2^2\beta_1^{d+1}=0 \qquad \text{if } m=2 \text{ and } k'=1,$$

$$2^{m-4+k}\beta_1^d\beta(1)+2^{m-4+2k}\beta_1^{d+1}=0 \qquad \text{if } m=2 \text{ and } k' \geqq 2,$$

$$2^{m-4+k}\beta_1^d\beta(1)-2^{m-4+2k}\beta_1^{d+1}=0 \qquad \text{if } m \geqq 3.$$

PROOF. By making use of (3.13) and Lemma 5.1, we have

$$(*) \qquad 2^{m-2}\beta_1^d\beta(1)^{k'} = \sum_{i=1}^{k'}\binom{k'}{i}2^{m-2+2i}\beta_1^{d+k-i}.$$

Thus, $(*)$ implies the desired results for $m \geqq 2$ and $k'=1$. Consider the case $k' \geqq 2$. Then the first term in the right hand side of $(*)$ is equal to

$$-k'2^{m-4+2k}\beta_1^{d+1}$$

by Lemmas 5.12 and 5.1–2, and the i–th term in $(*)$ is equal to

$$(-1)^{i-1}\binom{k'}{i}2^{m-4+2k}\beta_1^{d+1} \qquad (2 \leqq i \leqq k')$$

by Lemma 5.2. Therefore, we have

$$2^{m-2}\beta_1^d\beta(1)^{k'} = 2^{m-4+2k}\beta_1^{d+1}-k'2^{m-3+2k}\beta_1^{d+1} = (-1)^{k'}2^{m-4+2k}\beta_1^{d+1},$$

since $2^{m-2+2k}\beta_1^{d+1}=0$ by Lemma 5.1. On the other hand, we have

$$2^{m-2}\beta_1^d\beta(1)^{k'} = \begin{cases} (-1)^{k'-1}2^{k-2}\beta_1^d\beta(1) & \text{if } m=2, \\ (-1)^{k'}2^{m-4+k}\beta_1^d\beta(1) & \text{if } m \geqq 3, \end{cases}$$

by Lemmas 3.14, 5.10 and 5.2. Hence, we have the desired results. q.e.d.

LEMMA 6.6. *Suppose* $0 \leqq s \leqq m-3$, $k=2k'$ *and* d *is even under the assumption* (6.1). *Then*

$$\sum_{t=0}^{s+1}(-1)^{2^t}2^{m-s-4+2^t k}\beta_1^d\beta(s+1-t)=0.$$

PROOF. The desired relation follows immediately from Lemmas 6.2, 6.3 and 6.5. q.e.d.

LEMMA 6.7. *Suppose* $s=m-2 \geqq 0$, $k=2k'$ *and* d *is even under the assumption* (6.1). *Then*

$$\sum_{t=0}^{m-1}(-1)^{2^t}2^{2^t k-2}\beta_1^d\beta(m-1-t)=0 \quad \textit{if } k'=1,$$

$$\sum_{t=0}^{m-1}2^{2^t k-2}\beta_1^d\beta(m-1-t)=0 \quad \textit{if } k' \geqq 2.$$

PROOF. Lemmas 6.2, 6.4 and 6.5 imply the desired relation. q.e.d.

LEMMA 6.8. *Suppose* $1 \leqq s \leqq m-2$, $k=2k'+1$ *and* d *is even under the assumption* (6.1). *Then*

$$2^{m-s-2}\beta_1^d(\beta(s+1-t)^{2^{t-1}k}-\beta(s-t)^{2^t k})$$

is equal to

$$\begin{array}{ll}
2^{m-s-3+2^t k}\beta_1^d\beta(1)-3\ 2^{m-s-4+2^{t+1}k}\beta_1^{d+1} & \textit{if } k=3,\ s=t=1,\\
-2^{m-s-3+2^t k}\beta_1^d\beta(1)+2^{m-s-4+2^{t+1}k}\beta_1^{d+1} & \textit{if } k\geqq 5,\ s=t=1,\\
-2^{m-s-3+2^t k}\beta_1^d\beta(s)-7\ 2^{m-s-4+2^{t+1}k}\beta_1^d\beta(s-1) & \textit{if } k=3,\ s\geqq 2,\ t=1,\\
2^{m-s-3+2^t k}\beta_1^d\beta(s)+2^{m-s-4+2^{t+1}k}\beta_1^d\beta(s-1) & \textit{if } k\geqq 5,\ s\geqq 2,\ t=1,\\
2^{m-s-4+2^{t+1}k}\beta_1^d\beta(s-t) & \textit{if } k\geqq 3,\ 2\leqq t\leqq s-1,\\
\pm 2^{m-s-3+2^t k}\beta_1^d\beta(1)+2^{m-s-4+2^{t+1}k}\beta_1^{d+1} & \textit{if } s\geqq t\geqq 2,
\end{array}$$

where t *is an integer with* $1 \leqq t \leqq s$.

PROOF. Put $u=s-t$. By (3.13), we have

$$2^{m-s-2}\beta_1^d\beta(u+1)^{2^{t-1}k}=\sum_{i=0}^{2^t k'}\binom{2^t k'}{i}2^{m-s-2+2i}\beta_1^d\beta(u)^{2^{t+1}k'-i}\beta(u+1)^{2^{t-1}}.$$

The term for $i \geqq 3$ vanishes and the term for $i=2$ is equal to

$$\pm k' 2^{m-u+1}\beta_1^d\beta(u)^{2^{t+1}k'-2}\beta(u+1)^{2^{t-1}}$$

by Lemma 5.1. Also, by Lemmas 5.4 and 5.7,

$$2^{m-u}\beta_1^d\beta(u)^{2^{t+1}k'-1}\beta(u+1)^{2^{t-1}}+2^{m-u+1}\beta_1^d\beta(u)^{2^{t+1}k'-2}\beta(u+1)^{2^{t-1}}=0.$$

Therefore, we have

$$2^{m-s-2}\beta_1^d\beta(u+1)^{2^{t-1}k}=\sum_{i=0}^{2^{t-1}}\binom{2^{t-1}}{i}2^{m-s-2+2i}\beta_1^d\beta(u)^{2^t k-i},$$

and so

$$(*)\quad 2^{m-s-2}\beta_1^d(\beta(u+1)^{2^{t-1}k}-\beta(u)^{2^t k})=\sum_{i=1}^{2^{t-1}}\binom{2^{t-1}}{i}2^{m-s-2+2i}\beta_1^d\beta(u)^{2^t k-i}.$$

The i-th term in $(*)$ for $i \neq 1, 2, 4$ is equal to

$$(-1)^{i-1}\binom{2^{t-1}}{i}2^{m-s-4+2^{t+1}k}\beta_1^d\beta(u)$$

by Lemma 5.2. The i-th term in $(*)$ for $i=1$ $(t\geqq 1)$, $i=2$ $(t\geqq 2)$ and $i=4$ $(t\geqq 3)$ is equal to

$$\begin{aligned}(**)\ &\binom{2^{t-1}}{i}2^{m-s-2+2i}\beta_1^d\beta(u)^{4-i}(\beta(u+1)-4\beta(u))^{2^{t-1}k-2}\\ &=\binom{2^{t-1}}{i}2^{m-s-2+2i}\beta_1^d\beta(u)^{4-i}\{\beta(u+1)^{2^{t-1}k-2}+\textstyle\sum_{j=1}^{2^{t-1}k-2}(-1)^j\binom{2^{t-1}k-2}{j}2^{2j}\beta(u)^j\beta(u+1)^{2^{t-1}k-2-j}\}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\text{(by (3.13))}.\end{aligned}$$

In the case $i=1$, $(t, k)=(1, 3)$, $(**)$ is equal to

$$\binom{2^{t-1}}{i}2^{m-s-2+2i}\beta_1^d\beta(u)^{4-i}\beta(u+1)-\binom{2^{t-1}}{i}2^{m-s+2i}\beta_1^d\beta(u)^{5-i}.$$

In the case $i=1$, $t=1$, $k\geqq 5$ or $i=1, 2$, $t\geqq 2$, $k\geqq 3$, the j-th term in $(**)$ for $2\leqq j\leqq 2^{t-1}k-3$ vanishes by Lemma 5.1, and so $(**)$ is equal to

$$\begin{aligned}&\binom{2^{t-1}}{i}2^{m-s-2+2i}\beta_1^d\beta(u)^{4-i}\beta(u+1)^{2^{t-1}k-2}-(2^{t-1}k-2)\binom{2^{t-1}}{i}2^{m-s+2i}\beta_1^d\beta(u)^{5-i}\beta(u+1)^{2^{t-1}k-3}\\ &\qquad+(-1)^{2^{t-1}k}\binom{2^{t-1}}{i}2^{m-s-6+2i+2^tk}\beta_1^d\beta(u)^{2^{t-1}k+2-i}.\end{aligned}$$

In the case $i=4$, $t\geqq 3$, $k\geqq 3$, the j-th term in $(**)$ for $1\leqq j\leqq 2^{t-1}k-3$ vanishes by Lemma 5.1, and so $(**)$ is equal to

$$\binom{2^{t-1}}{i}2^{m-s-2+2i}\beta_1^d\beta(u)^{4-i}\beta(u+1)^{2^{t-1}k-2}+(-1)^{2^{t-1}k}\binom{2^{t-1}}{i}2^{m-s-6+2i+2^tk}\beta_1^d\beta(u)^{2^{t-1}k+2-i}.$$

Suppose $i=1, 2, 4$ and $1\leqq t\leqq s$. Then we have

$$\begin{aligned}&\binom{2^{t-1}}{i}2^{m-s-2+2i}\beta_1^d\beta(u)^{4-i}\beta(u+1)^{2^{t-1}k-2}\\ &=\begin{cases}(-1)^{2^{t-1}k+1}\binom{2^{t-1}}{i}2^{m-s-4+i+2^tk}\beta_1^d\beta(1) & \text{if } u=0 \text{ (by Lemmas 5.4, 5.2)},\\ (-1)^{2^{t-1}k+i+1}\binom{2^{t-1}}{i}2^{m-s-4+i+2^tk}\beta_1^d\beta(u+1) & \text{if } u\geqq 1 \text{ (by Lemmas 5.7, 5.2)}.\end{cases}\end{aligned}$$

Suppose $i=1$ and $(t, k)=(1, 3)$. Then

$$\binom{2^{t-1}}{i}2^{m-s+2i}\beta_1^d\beta(u)^{5-i}=\begin{cases}3\ 2^{m-s-4+2^{t+1}k}\beta_1^{d+1} & \text{if } u=0 \text{ (by Lemmas 5.12 and 5.2)},\\ 7\ 2^{m-s-4+2^{t+1}k}\beta_1^d\beta(u) & \text{if } u\geqq 1 \text{ (by Lemmas 5.12, 5.2 and 5.1)}.\end{cases}$$

In the case $i=1, 2, 4$, $(t, k)\neq(1, 3)$ and $u\geqq 0$,

$$(-1)^{2^{t-1}k}\binom{2^{t-1}}{i}2^{m-s-6+2i+2^tk}\beta_1^d\beta(u)^{2^{t-1}k+2-i}=(-1)^{i-1}\binom{2^{t-1}}{i}2^{m-s-4+2^{t+1}k}\beta_1^d\beta(u)\ \text{ (by Lemma 5.2)}.$$

In the case $i=1, 2$, $(t, k)\neq(1, 3)$ and $u\geqq 0$,

$$\begin{aligned}&(2^{t-1}k-2)\binom{2^{t-1}}{i}2^{m-s+2i}\beta_1^d\beta(u)^{5-i}\beta(u+1)^{2^{t-1}k-3}\\ &=\pm(2^{t-1}k-2)\binom{2^{t-1}}{i}2^{m-s+5+i}\beta_1^d\beta(u+1)^{2^{t-1}k-3}\ \text{(by Lemmas 5.1, 5.4, 5.7)},\\ &=\pm(2^{t-1}k-2)\binom{2^{t-1}}{i}2^{m-s-3+i+2^tk}\beta_1^d\beta(u+1)\ \text{ (by Lemma 5.2)},\end{aligned}$$

$$= \begin{cases} \pm 2^{m-s-2+2^t k}\beta_1^d\beta(s) & \text{if } i=1,\ t=1,\ k \geqq 5 \quad \text{(by Lemma 5.1)}, \\ 0 & \text{otherwise} \quad \text{(by Lemma 5.1)}. \end{cases}$$

Therefore, we have

$$(**) = \begin{cases} 2^{m-s-3+2k}\beta_1^d\beta(1) - 3\ 2^{m-s-4+2^2k}\beta_1^{d+1} & (u=0) \\ -2^{m-s-3+2k}\beta_1^d\beta(s) - 7\ 2^{m-s-4+2^2k}\beta_1^d\beta(s-1) & (u \geqq 1) \end{cases} \quad \text{if } i=1,\ (t,\ k)=(1,\ 3),$$

$$(**) = \begin{cases} -2^{m-s-3+2k}\beta_1^d\beta(1) + 2^{m-s-4+2^2k}\beta_1^{d+1} & (u=0) \\ 2^{m-s-3+2k}\beta_1^d\beta(s) + 2^{m-s-4+2^2k}\beta_1^d\beta(s-1) & (u \geqq 1) \end{cases} \quad \text{if } i=1,\ t=1 \text{ and } k \geqq 5,$$

$$(**) = \begin{cases} \binom{2^{t-1}}{i}\{-2^{m-s-4+i+2^t k}\beta_1^d\beta(1) + (-1)^{t-1}2^{m-s-4+2^{t+1}k}\beta_1^{d+1}\} & (u=0) \\ (-1)^{t-1}\binom{2^{t-1}}{i}\{2^{m-s-4+i+2^t k}\beta_1^d\beta(u+1) + 2^{m-s-4+2^{t+1}k}\beta_1^d\beta(u)\} & (u \geqq 1) \end{cases}$$

if $i=1,\ 2,\ t \geqq 2$ and $k \geqq 3$, and

$$(**) = (-1)^{t-1}\binom{2^{t-1}}{i}\{2^{m-s-4+i+2^t k}\beta_1^d\beta(u+1) + 2^{m-s-4+2^{t+1}k}\beta_1^d\beta(u)\} \qquad (u \geqq 0)$$

if $i=4,\ t \geqq 3$ and $k \geqq 3$.

Hence, we have the desired results by summarizing the i-th terms with $1 \leqq i \leqq 2^{t-1}$ in $(*)$. q.e.d.

LEMMA 6.9. *Suppose* $1 \leqq s \leqq m-3$, $k = 2k'+1$ *and* d *is even under the assumption* (6.1). *Then*

$$2^{m-s-2}\beta_1^d\beta(s)^k = -2^{m-s-4+k}\beta_1^d\beta(s+1) + 2^{m-s-4+2k}\beta_1^d\beta(s).$$

PROOF. By (3.13), we have

$$(*) \qquad 2^{m-s-1}\beta_1^d\beta(s)^k = \textstyle\sum_{i=0}^{k'}\binom{k'}{i}(-1)^i 2^{m-s-2+2i}\beta_1^d\beta(s)^{i+1}\beta(s+1)^{k'-i}$$

In the case $k'=1$, the right hand side of $(*)$ is equal to

$$-2^{m-s-1}\beta_1^d\beta(s+1) + 2^{m-s+2}\beta_1^d\beta(s)$$

by Lemma 5.7 and (3.13), and so the desired result is obtained

Suppose $k' \geqq 2$. Then the i-th term with $2 \leqq i \leqq k'-1$ in $(*)$ vanishes by Lemma 5.1, and so the right hand side of $(*)$ is equal to

$$2^{m-s-2}\beta_1^d\beta(s)\beta(s+1)^{k'} - k'\ 2^{m-s}\beta_1^d\beta(s)^2\beta(s+1)^{k'-1} + (-1)^{k'}2^{m-s-2+2k'}\beta_1^d\beta(s)^{k'+1}.$$

On the other hand

$$2^{m-s-2}\beta_1^d\beta(s)\beta(s+1)^{k'} = (-1)^{k'+1}2^{m-s-4+k}\beta_1^d\beta(s+1),$$

$$2^{m-s}\beta_1^d\beta(s)^2\beta(s+1)^{k'-1} = \pm 2^{m-s-3+k}\beta_1^d\beta(s+1)$$

by Lemmas 5.7, 5.11 and 5.1, and also

$$2^{m-s-2+2k'}\beta_1^d\beta(s)^{k'+1}=(-1)^{k'}2^{m-s-4+2k}\beta_1^d\beta(s)$$

by Lemmas 5.10 and 5.11. Therefore, we obtain the desired result from (*).

q.e.d.

LEMMA 6.10. *Suppose $s=m-2\geqq 1$, $k=2k'+1$ and d is even under the assumption* (6.1). *Then*

$$\beta_1^d\beta(m-2)^k=2^{k-2}\beta_1^d\beta(m-1)+2^{2k-2}\beta_1^d\beta(m-2).$$

PROOF. In the case $k'=1$, we have

$$\begin{aligned}\beta_1^d\beta(m-2)^k&=\beta_1^d\beta(m-2)\beta(m-1)-2^2\beta_1^d\beta(m-2)^2 \quad \text{(by (3.13))}\\ &=-2\beta_1^d\beta(m-1)-3\ 2^4\beta_1^d\beta(m-2) \quad \text{(by Lemmas 3.14 and 5.12)}\\ &=2\beta_1^d\beta(m-1)+2^4\beta_1^d\beta(m-2) \quad \text{(by Lemmas 5.1 and 5.12).}\end{aligned}$$

Thus, the desired result for $k'=1$ is obtained.

Suppose $k'\geqq 2$. Then we have

$$\beta_1^d\beta(m-2)^k=\beta_1^d\beta(m-2)\beta(m-1)^{k'}-k'2^2\beta_1^d\beta(m-2)^2\beta(m-1)^{k'-1}+(-1)^{k'}2^{2k'}\beta_1^d\beta(m-2)^{k'+1}$$

in the similar way to the proof of Lemma 6.9. Since $\beta(m-1)^2=-2^2\beta(m-1)$ and $\beta(m-2)\beta(m-1)=-2\beta(m-1)$ by Lemma 3.14, we have

$$\begin{aligned}&\beta_1^d\beta(m-2)\beta(m-1)^{k'}=(-1)^{k'}2^{k-2}\beta_1^d\beta(m-1),\\ &2^2\beta_1^d\beta(m-2)^2\beta(m-1)^{k'-1}=\pm 2^{k-1}\beta_1^d\beta(m-1) \quad \text{(by Lemma 5.1).}\end{aligned}$$

Therefore we obtain the desired result for $k'\geqq 2$. q.e.d.

LEMMA 6.11. *Suppose $1\leqq s\leqq m-3$, $k=2k'+1$ and d is even under the assumption* (6.1). *Then we have*

$$-3\ 2^{m-s-4+2^2k}\beta_1^{d+1}+2^{m-s-4+2k}\beta_1^d\beta(1)+2^{m-s-4+k}\beta_1^d\beta(2)=0 \quad \textit{if } s=1,\ k=3,$$

$$2^{m-s-4+2^3k}\beta_1^{d+1}+2^{m-s-4+2^2k}\beta_1^d\beta(1)+5\ 2^{m-s-4+2k}\beta_1^d\beta(2)+2^{m-s-4+k}\beta_1^d\beta(3)=0 \quad \textit{if } s=2,\ k=3,$$

$$\begin{aligned}&2^{m-s-4+2^{s+1}k}\beta_1^{d+1}+(1\pm 2^{s+1})2^{m-s-4+2^sk}\beta_1^d\beta(1)\\ &+\textstyle\sum_{t=2}^{s-2}2^{m-s-4+2^{t+1}k}\beta_1^d\beta(s-t)-7\ 2^{m-s-4+2^2k}\beta_1^d\beta(s-1)\\ &+5\ 2^{m-s-4+2k}\beta_1^d\beta(s)+2^{m-s-4+k}\beta_1^d\beta(s+1)=0 \quad \textit{if } s\geqq 3,\ k=3,\\ &2^{m-s-4+2^{s+1}k}\beta_1^{d+1}+(1\pm 2^{s+1})2^{m-s-4+2^sk}\beta_1^d\beta(1)\\ &+\textstyle\sum_{t=-1}^{s-2}2^{m-s-4+2^{t+1}k}\beta_1^d\beta(s-t)=0 \quad \textit{if } s\geqq 1,\ k\geqq 5.\end{aligned}$$

PROOF. The desired results follow immediately from Lemmas 6.8, 6.9 and 5.1.

q.e.d.

LEMMA 6.12. *Suppose $s=m-2\geqq 1$, $k=2k'+1$ and d is even under the assumption*

(6.1). *Then, we have*

$$-3\ 2^{2^2k-2}\beta_1^{d+1}+2^{2k-2}\beta_1^d\beta(1)-2^{k-1}\beta_1^d\beta(2)=0 \quad \text{if } m=3,\ k=3,$$

$$2^{2^3k-2}\beta_1^{d+1}+2^{2^2k-2}\beta_1^d\beta(1)+5\ 2^{2k-2}\beta_1^d\beta(2)-2^{k-2}\beta_1^d\beta(3)=0 \quad \text{if } m=4,\ k=3,$$

$$2^{2^{m-1}k-2}\beta_1^{d+1}+(1\pm 2^{m-1})2^{2^{m-2}k-2}\beta_1^d\beta(1)+\textstyle\sum_{t=2}^{m-4}2^{2^{t+1}k-2}\beta_1^d\beta(m-2-t)$$

$$-7\ 2^{2^2k-2}\beta_1^d\beta(m-3)+5\ 2^{2k-2}\beta_1^d\beta(m-2)-2^{k-2}\beta_1^d\beta(m-1)=0 \quad \text{if } m\geqq 5,\ k=3,$$

$$2^{2^{m-1}k-2}\beta_1^{d+1}+(1\pm 2^{m-1})2^{2^{m-2}k-2}\beta_1^d\beta(1)$$

$$+\textstyle\sum_{t=-1}^{m-4}(-1)^{2^{t+1}}2^{2^{t+1}k-2}\beta_1^d\beta(m-2-t)=0 \quad \text{if } m\geqq 3,\ k\geqq 5.$$

PROOF. The desired results follow immediately from Lemmas 6.8 and 6.10. q.e.d.

LEMMA 6.13. *Suppose* $1\leqq s\leqq m-2$, $k=2k'+1$ *and* d *is even under the assumption* (6.1). *Then,* $2^{m-s-2}\beta_1^d\beta(s)^k$ *is equal to*

$$2^{m-s-3+2k}\beta_1^d\beta(1)-3\ 2^{m-s-4+2^2k}\beta_1^{d+1} \quad \text{if } s=1,\ k=3,$$

$$-2^{m-s-3+2k}\beta_1^d\beta(2)+2^{m-s-4+2^2k}\beta_1^d\beta(1)+2^{m-s-4+2^3k}\beta_1^{d+1} \quad \text{if } s=2,\ k=3,$$

$$-2^{m-s-3+2k}\beta_1^d\beta(s)-7\ 2^{m-s-4+2^2k}\beta_1^d\beta(s-1)$$

$$+\textstyle\sum_{t=2}^{s-2}2^{m-s-4+2^{t+1}k}\beta_1^d\beta(s-t)+(1\pm 2^{s+1})2^{m-s-4+2^sk}\beta_1^d\beta(1)$$

$$+2^{m-s-4+2^{s+1}k}\beta_1^{d+1} \quad \text{if } s\geqq 3,\ k=3,$$

$$-2^{m-s-3+2k}\beta_1^d\beta(1)+2^{m-s-4+2^2k}\beta_1^{d+1} \quad \text{if } s=1,\ k\geqq 5,$$

$$2^{m-s-3+2k}\beta_1^d\beta(s)+\textstyle\sum_{t=1}^{s-2}2^{m-s-4+2^{t+1}k}\beta_1^d\beta(s-t)$$

$$+(1\pm 2^{s+1})2^{m-s-4+2^sk}\beta_1^d\beta(1)+2^{m-s-4+2^{s+1}k}\beta_1^{d+1} \quad \text{if } s\geqq 2,\ k\geqq 5.$$

PROOF. The desired results follow from Lemmas 6.5 and 5.1. q.e.d.

LEMMA 6.14. *Suppose* $2\leqq s\leqq m-2$ *and* d *is even under the assumption* (6.1). *Then*

$$2^{m-s-2}\beta_1^d\beta(s-1)\beta(s)^k=\pm\varepsilon(k)2^{m-s-2+2k}\beta_1^d\beta(s)-2^{m-s-2+2^2k}\beta_1^d\beta(s-1),$$

where $\varepsilon(k)=0$ *if* k *is even*, $=1$ *if* k *is odd.*

PROOF. By Lemma 6.2, we have

$$2^{m-s-2}\beta_1^d\beta(s-1)\beta(s)^k=\textstyle\sum_{t=1}^{s}2^{m-s-4+2^{t+1}k}\beta_1^d\beta(s-t)\beta(s-1)$$

if k is even. On the other hand,

$$(*)\qquad 2^{m-s-4+2^{t+1}k}\beta_1^d\beta(s-t)\beta(s-1)=\begin{cases}0 & \text{if } 2\leqq t\leqq s,\\ -2^{m-s-2+2^2k}\beta_1^d\beta(s-1) & \text{if } t=1,\end{cases}$$

for any $k\geqq 2$ by Lemma 5.1 and (3.13). Therefore, the desired result for even k follows. Let k be odd. Then, by Lemmas 5.7 and 5.1, we have

$$2^{m-s-3+2k}\beta_1^d\beta(s-1)\beta(s) = \pm 2^{m-s-2+2k}\beta_1^d\beta(s).$$

Thus the desired result for odd k follows from Lemmas 6.13, 5.1 and (*) above.

q.e.d.

Lemma 6.15. *Suppose $2 \leqq s \leqq m-2$ and $d \geqq 2$ is even under the assumption* (6.1). *Then*

$$2^{m-s-3+k}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))$$

$$= \begin{cases} (-1)^{k-1}2^{m-s-2}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1} \pm \\ \qquad 2^{m-s-2}\beta_1^{d-1}\beta(1)(2+\beta(0))\prod_{t=1}^{s-1}\beta(t)\beta(s)^{k-2}\beta(s+1) & \text{if } s \leqq m-3, \\ (-1)^{k-1}2^{m-s-2}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1} & \text{if } s = m-2. \end{cases}$$

Proof. By Lemma 3.14, we have

$$2^{k-l-2}\beta_1^{d-1}\beta(s)^l P_{m,1} = 0 \text{ for } 0 \leqq l \leqq k-2,$$

and so

(*) $2^{m-s-3+k-l}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^l +$

$$\sum_{I_s} 2^{m-s-3+k-l-j}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^l\beta(i_1)\cdots\beta(i_j) = 0.$$

In the case $s = m-2$, (*) is equal to the relation

$$2^{m-s-3+k-l}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^l = -2^{m-s-4+k-l}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{l+1}$$

for $0 \leqq l \leqq k-2$. Thus, we have the desired relation for $s = m-2$. Consider the case $s \leqq m-3$. By Lemma 5.1, the terms for $(i_1, \cdots, i_j) \in I_s$ vanish except for (s), $(s+1)$ and $(s, s+1)$. The term for $(s+1)$ is equal to

$$2^{m-s-4+k-l}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^l\beta(s+1)$$

$$= \sum_{i=0}^{s-1} \pm 2^{m-s-3+k-l}\beta_1^{d-1}\beta(1)\beta(0)\cdots\widehat{\beta(i)}\cdots\beta(s-1)\beta(s)^l\beta(s+1)$$

$$\pm 2^{m-s-4+k-l}\beta_1^{d-1}\beta(1)\beta(0)\cdots\beta(s-1)\beta(s)^l\beta(s+1) \text{ (by Lemma 5.1)},$$

where the notation $\widehat{\beta(i)}$ means that $\beta(i)$ is deleted. The term for $(s, s+1)$ is equal to

$$2^{m-s-5+k-l}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{l+1}\beta(s+1)$$

$$= \sum_{i=0}^{s-1} \pm 2^{m-s-4+k-l}\beta_1^{d-1}\beta(1)\beta(0)\cdots\widehat{\beta(i)}\cdots\beta(s-1)\beta(s)^{l+1}\beta(s+1)$$

$$\pm 2^{m-s-5+k-l}\beta_1^{d-1}\beta(1)\beta(0)\cdots\beta(s-1)\beta(s)^{l+1}\beta(s+1) \text{ (by Lemma 5.1)}.$$

On the other hand, by Lemma 5.7, we have

$$2^{m-s-4+k-l}\beta_1^{d-1}\beta(1)\beta(0)\cdots\widehat{\beta(i)}\cdots\beta(s-1)\beta(s)^l(2+\beta(s))\beta(s+1) = 0 \text{ if } k-l \geqq 3 \text{ or } i \geqq 1,$$

$$2^{m-s-5+k-1}\beta_1^{d-1}\beta(1)\beta(0)\cdots\beta(s-1)\beta(s)^l(2+\beta(s))\beta(s+1) = 0 \text{ if } k-l \geqq 3.$$

Also, if $k-l = 2$, we have

$$2^{m-s-5+k-l}\beta_1^{d-1}\beta(1)\beta(0)\cdots\beta(s-1)\beta(s)^{l+1}\beta(s+1) = 0 \text{ (by Lemma 5.3)},$$

$$2^{m-s-4+k-l}\beta_1^{d-1}\beta(1)\prod_{t=1}^{s-1}\beta(t)\beta(s)^{k-1}\beta(s+1) = 2^{m-s-2}\beta_1^{d+2^s k-1}\beta(s+1) = 0 \text{ (by Lemma 5.1)},$$

since $\beta(t)=\beta_1^{2^t}+2^2Q(\beta_1)$ by the definition of $\beta(t)$ in (3.13), where $Q(\beta_1)$ is a polynomial in β_1 whose constant term is zero. Therefore, we have the following relations by (*)

$$2^{m-s-3+k-l}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^l = -2^{m-s-4+k-l}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{l+1}$$

for $0\leqq l\leqq k-3$, and

$$2^{m-s-1}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-2}$$

$$= -2^{m-s-2}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1} \pm 2^{m-s-2}\beta_1^{d-1}\beta(1)(2+\beta(0))\prod_{t=1}^{s-1}\beta(t)\beta(s)^{k-2}\beta(s+1).$$

The desired relation for $s\leqq m-3$ follows immediately from these relations. q.e.d.

LEMMA 6.16. *Under the same assumption as in Lemma* 6.15, *we have*

$$2^{m-s-2}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1} = 2^{m-s-2}\beta_1^{d+1}\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1} \pm$$
$$2^{m-s-1}\beta_1^{d}\prod_{t=0}^{s-2}\beta(t)(2+\beta(s-1))\beta(s)^{k}.$$

PROOF. Since $\beta(1)=\beta_1^2+2^2\beta_1$ by (3.13),

$$2^{m-s-2}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1} = 2^{m-s-2}\beta_1^{d+1}\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1} +$$
$$2^{m-s}\beta_1^{d}\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1}.$$

Also, $2\beta_1^d\prod_{t=0}^{s-2}(2+\beta(t))\beta(s)^{k-2}P_{m,s}=0$ by Lemma 3.14, and so

$$2^{m-s}\beta_1^d\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1} + \sum_{I_s}2^{m-s-j}\beta_1^d\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1}\beta(i_1)\cdots\beta(i_j)=0.$$

The terms in $\sum_{I_s}$ vanish except for the term for $(s)\in I_s$ by Lemma 5.1. The term for (s) is equal to

$$2^{m-s-1}\beta_1^d\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^k = \pm 2^{m-s-1}\beta_1^d\prod_{t=0}^{s-2}\beta(t)(2+\beta(s-1))\beta(s)^k$$

by making use of Lemmas 5.7 and 5.1. Therefore, we have the desired result. q.e.d.

LEMMA 6.17. *Under the same assumption as in Lemma* 6.15, *we have*

$$2^{m-s-1}\beta_1^d\prod_{t=0}^{s-2}\beta(t)(2+\beta(s-1))\beta(s)^k = \pm 2^{m-s-1+2k}\beta_1^{d-1}\beta(s),$$

$$2^{m-s-2}\beta_1^{d-1}\beta(1)(2+\beta(0))\prod_{t=1}^{s-1}\beta(t)\beta(s)^{k-2}\beta(s+1) = \pm 2^{m-s-1+2k}\beta_1^{d-1}\beta(s)\pm 2^{m-s-2+2k}\beta_1^d\beta(s).$$

PROOF. Since $\beta(t)^2=\beta(t+1)-2^2\beta(t)$ by (3.13), the left hand side of the first relation is equal to

$$\pm 2^{m-s-1}\beta_1^{d-1}\beta(s)^{k+1}\pm 2^{m-s}\beta_1^{d-1}\beta(s-1)\beta(s)^k$$

by Lemma 5.1. On the other hand

$$2^{m-s-1}\beta_1^{d-1}\beta(s)^{k+1} = 2^{m-s-1}\beta_1^{d-1}\sum_{i=0}^{k+1}\binom{k+1}{i}2^{2i}\beta(s-1)^{2k+2-i}=0$$

by (3.13) and Lemma 5.1. Also, we have

$$2^{m-s}\beta_1^{d-1}\beta(s-1)\beta(s)^k = \pm 2^{m-s+1}\beta_1^{d-1}\beta(s)^k = \pm 2^{m-s-1+2k}\beta_1^{d-1}\beta(s)$$

by Lemmas 5.7, 5.1 and 5.2. Thus we obtain the first relation. The left hand side of the second relation is equal to

$$(*)\quad \pm 2^{m-s-1}\beta_1^{d-1}\beta(1)\Pi_{t=1}^{s-1}\beta(t)\beta(s)^k \pm 2^{m-s-2}\beta_1^d\beta(1)\Pi_{t=1}^{s-1}\beta(t)\beta(s)^k$$
$$\pm 2^{m-s+1}\beta_1^{d-1}\beta(1)\Pi_{t=1}^{s-1}\beta(t)\beta(s)^{k-1} \pm 2^{m-s}\beta_1^d\beta(1)\Pi_{t=1}^{s-1}\beta(t)\beta(s)^{k-1}$$

by (3.13) and Lemma 5.1. The first term of $(*)$ is equal to $2^{m-s-1}\beta_1^{d-1}\beta(s)^{k+1}$ by (3.13) and Lemma 5.1, and this is equal to zero, as is shown in the proof of the first relation. The second term of $(*)$ is equal to $2^{m-s-2}\beta_1^d\beta(s)^{k+1}$ by (3.13) and Lemma 5.1, and is equal to zero by Lemma 5.3. The third term of $(*)$ is equal to $2^{m-s+1}\beta_1^{d-1}\beta(s)^k$ by (3.13) and Lemma 5.1, and

$$2^{m-s+1}\beta_1^{d-1}\beta(s)^k = \pm 2^{m-s-1+2k}\beta_1^{d-1}\beta(s)$$

by Lemma 5.7. The last term of $(*)$ is equal to $2^{m-s}\beta_1^d\beta(s)^k$ by (3.13) and Lemma 5.1, and

$$2^{m-s}\beta_1^d\beta(s)^k = \pm 2^{m-s-2+2k}\beta_1^d\beta(s)$$

by Lemma 5.2. Therefore we have the second relation. q.e.d.

By Lemmas 6.15–17, we see easily the following

Lemma 6.18. *Under the same assumption as in Lemma 6.15, we have*

$$2^{m-s-3+k}\beta_1^{d-1}\beta(1)\Pi_{t=0}^{s-1}(2+\beta(t))$$
$$=\begin{cases}(-1)^{k-1}2^{m-s-2}\beta_1^{d+1}\Pi_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1} \pm 2^{m-s-2+2k}\beta_1^d\beta(s) & \text{if } s\leqq m-3,\\ (-1)^{k-1}2^{m-s-2}\beta_1^{d+1}\Pi_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1} \pm 2^{m-s-1+2k}\beta_1^{d-1}\beta(s) & \text{if } s=m-2.\end{cases}$$

Lemma 6.19. *Under the same assumption as in Lemma 6.15, we have*

$$2^{m-s-1}\beta_1^d\beta(s-1)\beta(s)^{k-1} = (-1)^k 2^{m-s-4+2k}\beta_1^d\beta(s) - 2^{m-s-2}\beta_1^d\beta(s-1)\beta(s)^k.$$

Proof. By Lemma 3.14, $\beta_1^d\beta(s)^{k-2}P_{m,s}=0$, and so

$$2^{m-s-1}\beta_1^d\beta(s-1)\beta(s)^{k-1} + 2^{m-s}\beta_1^d\beta(s)^{k-1} +$$
$$\textstyle\sum_{I_s} 2^{m-s-1-j}\beta_1^d(2+\beta(s-1))\beta(s)^{k-1}\beta(i_1)\cdots\beta(i_j)=0.$$

The second term is equal to

$$3\ 2^{m-s+2}\beta_1^d\beta(s)^{k-2} \quad \text{(by Lemma 5.11)}$$
$$=(-1)^{k-1}3\ 2^{m-s-4+2k}\beta_1^d\beta(s) \quad \text{(by Lemma 5.2)}.$$

The terms for $(i_1, \cdots, i_j) \in I_s$ vanish except for (s) by Lemma 5.1 and (3.13). The term for (s) is equal to

$$2^{m-s-2}\beta_1^d(2+\beta(s-1))\beta(s)^k = (-1)^k 2^{m-s-3+2k}\beta_1^d\beta(s) + 2^{m-s-2}\beta_1^d\beta(s-1)\beta(s)^k \quad \text{(by Lemma 5.11)}.$$

These imply the desired result. q.e.d.

LEMMA 6.20. *Under the same assumption as in Lemma* 6.15, *we have*

$$2^{m-s-2}\beta_1^d\beta(s)^k$$
$$= 2^{m-s-2}\beta_1^{d+1}\Pi_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1} + (-1)^k 2^{m-s-4+2k}\beta_1^d\beta(s)$$
$$- 2^{m-s-2}\beta_1^d\beta(s-1)\beta(s)^k + \textstyle\sum_{u=0}^{s-2} 2^{m-s-1}\beta_1^d\beta(u)\Pi_{t=u+1}^{s-1}(2+\beta(t))\beta(s)^{k-1}.$$

PROOF. By Lemma 3.14(i),

$$\beta(s) = \beta_1\Pi_{t=0}^{s-1}(2+\beta(t)) + 2\textstyle\sum_{u=0}^{s-1}\beta(\mathrm{u})\Pi_{t=u+1}^{s-1}(2+\beta(t)).$$

Hence, we have

$$2^{m-s-2}\beta_1^d\beta(s)^k = 2^{m-s-2}\beta_1^{d+1}\Pi_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1} + 2^{m-s-1}\beta_1^d\beta(s-1)\beta(s)^{k-1}$$
$$+ \textstyle\sum_{u=0}^{s-2} 2^{m-s-1}\beta_1^d\beta(\mathrm{u})\Pi_{t=u+1}^{s-1}(2+\beta(t))\beta(s)^{k-1}.$$

Therefore, the desired result follows from Lemma 6.19. q.e.d.

LEMMA 6.21. *Under the same assumption as in Lemma* 6.15, *we have*

$$2^{m-s+1}\beta_1^{d-1}\Pi_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1} = 0.$$

PROOF. By Lemma 3.14, $2^2\beta_1^{d-1}\Pi_{t=0}^{s-2}(2+\beta(t))\beta(s)^{k-2}P_{m,s} = 0$, and so

$$2^{m-s+1}\beta_1^{d-1}\Pi_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1} + \textstyle\sum_{I_s} 2^{m-s+1-j}\beta_1^{d-1}\Pi_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1}\beta(i_1)\cdots\beta(i_j) = 0.$$

The terms for $(i_1, \cdots, i_j) \in I_s$ vanish except for (s) by Lemma 5.1. The term for (s) is equal to

$$2^{m-s}\beta_1^{d-1}\Pi_{t=0}^{s-1}(2+\beta(t))\beta(s)^k = 0 \text{ (by Lemma 5.7).}$$

Thus, we have the desired result. q.e.d.

LEMMA 6.22. *Under the same assumption as in Lemma* 6.15, *we have*

$$2^{m-s}\beta_1^d\Pi_{t=1}^{s-1}(2+\beta(t))\beta(s)^{k-1} = 0.$$

PROOF. By Lemma 3.14, $2\beta_1^d\Pi_{t=1}^{s-1}(2+\beta(t))\beta(s)^{k-2}P_{m,s} = 0$, and so

$$2^{m-s}\beta_1^d\Pi_{t=1}^{s-1}(2+\beta(t))\beta(s)^{k-1} + \textstyle\sum_{I_s} 2^{m-s-j}\beta_1^d\Pi_{t=1}^{s-1}(2+\beta(t))\beta(s)^{k-1}\beta(i_1)\cdots\beta(i_j) = 0.$$

The terms for $(i_1, \cdots, i_j) \in I_s$ vanish except for (s). The term for (s) is equal to

$$2^{m-s-1}\beta_1^d\Pi_{t=1}^{s-1}(2+\beta(t))\beta(s)^k = 0 \text{ (by Lemma 5.7).}$$

This implies the desired result. q.e.d.

LEMMA 6.23. *Under the same assumption as in Lemma* 6.15, *we have*

$$2^{m-s-1}\beta_1^{d-2}\beta(1)\Pi_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1}$$
$$= \begin{cases} (-1)^{k+1}2^{m-s-2+k}\beta_1^{d-2}\beta(1)\Pi_{t=0}^{s-1}(2+\beta(t)) \pm 2^{m-s-1+2k}\beta_1^{d-1}\beta(s) & \text{if } s \leqq m-3, \\ (-1)^{k+1}2^{m-s-2+k}\beta_1^{d-2}\beta(1)\Pi_{t=0}^{s-1}(2+\beta(t)) & \text{if } s = m-2. \end{cases}$$

PROOF. By Lemma 3.14, $2^{k-l-1}\beta_1^{d-2}\beta(s)^l P_{m,1}=0$ for $0\leqq l\leqq k-2$, and so

$$2^{m-s+k-l-2}\beta_1^{d-2}\beta(1)\textstyle\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^l$$
$$+\textstyle\sum_{I_s}2^{m-s+k-l-2-j}\beta_1^{d-2}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^l\beta(i_1)\cdots\beta(i_j)=0.$$

The terms for $(i_1,\cdots,i_j)\in I_s$ vanish except for (s), $(s+1)$ and $(s, s+1)$ by Lemma 5.1. Here we notice that the terms for $(s+1)$ and $(s, s+1)$ appear in $\sum_{I_s}$ only for the case $s\leqq m-3$. In the case $2\leqq s\leqq m-3$, the sum of the terms for $(s+1)$ and $(s, s+1)$ in $\sum_{I_s}$ is equal to

$$(*)\qquad \pm 2^{m-s-4+k-l}\beta_1^{d-2}\beta(1)\textstyle\prod_{t=0}^{s-1}\beta(t)\beta(s)^l(2\pm\beta(s))\beta(s+1)$$

by Lemma 5.1. By Lemma 5.7, $(*)=0$ if $0\leqq l\leqq k-3$. Suppose $l=k-2$. Then $(*)$ is equal to

$$\pm 2^{m-s-1}\beta_1^{d-1}\beta(s)^{k-1}\beta(s+1)\pm 2^{m-s-2}\beta_1^{d-1}\beta(s)^k\beta(s+1)$$
$$=\pm 2^{m-s+1}\beta_1^{d-1}\beta(s)^k\pm 2^{m-s-1}\beta_1^{d-1}\beta(s)^{k+1}$$

by (3.13) and Lemma 5.1. The term $2^{m-s-1}\beta_1^{d-1}\beta(s)^{k+1}$ vanishes as is shown in the first half of the proof of Lemma 6.17. Hence, we have

$$(*)=\begin{cases}0 & \text{if } 0\leqq l\leqq k-3,\\ \pm 2^{m-s-1+2k}\beta_1^{d-1}\beta(s) & \text{if } l=k-2,\end{cases}$$

and so

$$2^{m-s+k-l-2}\beta_1^{d-2}\beta(1)\textstyle\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^l$$
$$=\begin{cases}-2^{m-s+k-l-3}\beta_1^{d-2}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{l+1}\\ \qquad\qquad \text{if } 0\leqq l\leqq k-2\ (s=m-2) \text{ or } 0\leqq l\leqq k-3\ (s\leqq m-3),\\ -2^{m-s+k-l-3}\beta_1^{d-2}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{l+1}\pm 2^{m-s-1+2k}\beta_1^{d-1}\beta(s) \text{ if } l=k-2\ (s\leqq m-3).\end{cases}$$

This implies the desired results. q.e.d.

LEMMA 6.24. *Under the same assumption as in Lemma* 6.15, *we have*

$$2^{m-s-1}\beta_1^{d+1}\textstyle\prod_{t=1}^{s-1}(2+\beta(t))\beta(s)^{k-1}$$
$$=\begin{cases}(-1)^{k+1}2^{m-s-2+k}\beta_1^{d-2}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\pm 2^{m-s-1+2k}\beta_1^{d-1}\beta(s) & \textit{if } 2\leqq s\leqq m-3,\\ (-1)^{k+1}2^{m-s-2+k}\beta_1^{d-2}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t)) & \textit{if } s=m-2.\end{cases}$$

PROOF. By (3.13), we have

$$2^{m-s-1}\beta_1^{d+1}\textstyle\prod_{t=1}^{s-1}(2+\beta(t))\beta(s)^{k-1}$$
$$=2^{m-s-1}\beta_1^{d-2}\beta(1)\textstyle\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1}-2^{m-s+1}\beta_1^{d-1}\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1}$$
$$-2^{m-s}\beta_1^{d}\textstyle\prod_{t=1}^{s-1}(2+\beta(t))\beta(s)^{k-1}.$$

Therefore, the desired result follows from Lemmas 6.21–23. q.e.d.

LEMMA 6.25. *Suppose* $3 \leqq s \leqq m-2$, $1 \leqq u \leqq s-2$ *and* $d \geqq 2$ *is even under the assumption* (6.1). *Then*

$$2^{m-s-1}\beta_1^d\beta(u)\prod_{t=u+1}^{s-1}(2+\beta(t))\beta(s)^{k-1} = -2^{m-s-2}\beta_1^d\beta(u)\prod_{t=u+1}^{s-1}(2+\beta(t))\beta(s)^k.$$

PROOF. By Lemma 3.14, $\beta_1^d\beta(u)\prod_{t=u+1}^{s-2}(2+\beta(t))\beta(s)^{k-2}P_{m,s} = 0$, and so

$$2^{m-s-1}\beta_1^d\beta(u)\prod_{t=u+1}^{s-1}(2+\beta(t))\beta(s)^{k-1}$$
$$+\sum_{I_s} 2^{m-s-1-j}\beta_1^d\beta(u)\prod_{t=u+1}^{s-1}(2+\beta(t))\beta(s)^{k-1}\beta(i_1)\cdots\beta(i_j) = 0.$$

The terms for $(i_1, \cdots, i_j) \in I_s$ vanish except for (s), $(s+1)$ and $(s, s+1)$ by Lemma 5.1. The term for $(s+1)$ is equal to

$$2^{m-s-2}\beta_1^d\beta(u)\prod_{t=u+1}^{s-1}(2+\beta(t))\beta(s)^{k-1}\beta(s+1)$$
$$= 2^{m-s-2}\beta_1^d\prod_{t=u}^{s-1}\beta(t)\beta(s)^{k-1}\beta(s+1) \quad \text{(by Lemma 5.1)}$$
$$= 2^{m-s-2}\beta_1^d\prod_{t=u}^{s-1}\beta(t)\beta(s)^{k+1} = 0 \quad \text{(by (3.13) and Lemma 5.1).}$$

The term for $(s, s+1)$ is equal to

$$2^{m-s-3}\beta_1^d\beta(u)\prod_{t=u+1}^{s-1}(2+\beta(t))\beta(s)^k\beta(s+1)$$
$$= 2^{m-s-3}\beta_1^d\prod_{t=u}^{s-1}\beta(t)\beta(s)^k\beta(s+1) \quad \text{(by Lemma 5.1)}$$
$$= 2^{m-s-3}\beta_1^d\prod_{t=u}^{s-1}\beta(t)\beta(s)^{k+2} = 0 \quad \text{(by (3.13) and Lemma 5.1).}$$

Therefore, we have the desired result. q.e.d.

LEMMA 6.26. *Under the same assumption as in Lemma* 6.25, *we have*

$$2^{m-s-2}\beta_1^d\beta(u)\prod_{t=u+1}^{s-1}(2+\beta(t))\beta(s)^k$$
$$= \sum_{l=1}^{s}(-1)^{2^{l-1}}2^{m-s-3+2^{l+1}k}\beta_1^d\beta(u)\prod_{t=u+1}^{s-2}(2+\beta(t))\beta(s-l).$$

PROOF. Since

$$2^{m-s-2}\beta_1^d\beta(u)\prod_{t=u+1}^{s-1}(2+\beta(t))\beta(s)^k = 2^{m-s-1}\beta_1^d\beta(u)\prod_{t=u+1}^{s-2}(2+\beta(t))\beta(s)^k$$
$$+ 2^{m-s-2}\beta_1^d\beta(u)\prod_{t=u+1}^{s-2}(2+\beta(t))\beta(s-1)\beta(s)^k,$$

the desired result for even k follows from Lemmas 6.2 and 6.14, and also the one for odd k follows from Lemmas 6.13 and 6.14 by making use of Lemma 5.1. q.e.d.

LEMMA 6.27. *Suppose* $2 \leqq s \leqq m-2$ *and* $d \geqq 2$ *is even under the assumption* (6.1). *Then*

$$2^{m-s-1}\beta_1^d\sum_{u=0}^{s-2}\beta(u)\prod_{t=u+1}^{s-1}(2+\beta(t))\beta(s)^{k-1}$$

$$
= \begin{cases}
-\sum_{u=1}^{s-2}\sum_{l=1}^{s}(-1)^{2^{l-1}}2^{m-s-3+2^{l+1}k}\beta_1^d\beta(u)\prod_{t=u+1}^{s-2}(2+\beta(t))\beta(s-l) \\
+(-1)^{k+1}2^{m-s-2+k}\beta_1^{d-2}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t)) \pm 2^{m-s-1+2k}\beta_1^{d-1}\beta(s) \quad \text{if } s \leqq m-3, \\
-\sum_{u=1}^{s-2}\sum_{l=1}^{s}(-1)^{2^{l-1}}2^{m-s-3+2^{l+1}k}\beta_1^d\beta(u)\prod_{t=u+1}^{s-2}(2+\beta(t))\beta(s-l) \\
+(-1)^{k+1}2^{m-s-2+k}\beta_1^{d-2}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t)) \quad \text{if } s = m-2.
\end{cases}
$$

PROOF. The lemma is the immediate consequence of Lemmas 6.24–26.

q.e.d.

LEMMA 6.28. *Under the same assumption as in Lemma* 6.25, *we have*

$$
\sum_{l=1}^{s}\sum_{u=1}^{s-2}(-1)^{2^{l-1}}2^{m-s-3+2^{l+1}k}\beta_1^d\beta(u)\beta(s-l)\prod_{t=u+1}^{s-2}(2+\beta(t))
$$
$$
= 2^{m-s-2+2^2k}\beta_1^d\beta(s-1) - 2^{m-4+2^sk}\beta_1^d\beta(1).
$$

PROOF. Since $2^{m-s-3+2^2k}\beta_1^d(2+\beta(s-2))\beta(s-1)=0$ by Lemma 5.9, the term for $l=1$ in $\sum_{l=1}^{s}$ is equal to

$$
2^{m-s-2+2^2k}\beta_1^d\beta(s-1).
$$

Consider the terms for $3 \leqq l \leqq s$ in $\sum_{l=1}^{s}$. Then

$$
2^{m-s-3+2^{l+1}k}\beta_1^d\beta(s-l)\beta(u)=0
$$

for any u with $s-l \leqq u \leqq s-2$ by Lemma 5.1. Hence, the term for $l\,(3 \leqq l \leqq s)$ is equal to

$$
\sum_{u=1}^{s-l}2^{m-s-3+2^{l+1}k}\beta_1^d\beta(u)\beta(s-l)\prod_{t=u+1}^{s-2}(2+\beta(t)).
$$

Therefore, the summation $\sum_{l=2}^{s}$ of the left hand side of the desired relation is equal to

$$
(*) \qquad \sum_{l=2}^{s-1}\sum_{u=1}^{s-l}2^{m-s-3+2^{l+1}k}\beta_1^d\beta(u)\beta(s-l)\prod_{t=u+1}^{s-2}(2+\beta(t)).
$$

Also, by Lemma 5.1

$$
2^{m-s-3+2^{l+1}k}\beta_1^d\beta(u)\beta(s-l)\beta(s-i)=0
$$

for any i, u with $2 \leqq i \leqq l-1$, $1 \leqq u \leqq s-l$. Hence

$$
\sum_{u=1}^{s-l}2^{m-s-3+2^{l+1}k}\beta_1^d\beta(u)\beta(s-l)\prod_{t=u+1}^{s-2}(2+\beta(t))
$$
$$
= \sum_{u=1}^{s-l}2^{m-s+l-5+2^{l+1}k}\beta_1^d\beta(u)\beta(s-l)\prod_{t=u+1}^{s-l}(2+\beta(t))
$$

for $2 \leqq l \leqq s-1$. Therefore, $(*)$ is equal to

$$
\sum_{l=2}^{s-1}\{2^{m-s+l-5+2^{l+1}k}\beta_1^d\beta(s-l)^2 +
$$
$$
\sum_{u=1}^{s-l-1}2^{m-s+l-5+2^{l+1}k}\beta_1^d\beta(u)\beta(s-l)(2+\beta(s-l))\prod_{t=u+1}^{s-l-1}(2+\beta(t))\}.
$$

On the other hand, by Lemma 5.1

$$
2^{m-s+l-5+2^{l+1}k}\beta_1^d\beta(s-l+1)=0,
$$

and so $(*)$ is equal to

$$
-\sum_{l=2}^{s-1}\{2^{m-s+l-3+2^{l+1}k}\beta_1^d\beta(s-l) + \sum_{u=1}^{s-l-1}2^{m-s+l-4+2^{l+1}k}\beta_1^d\beta(u)\beta(s-l)\prod_{t=u+1}^{s-l-1}(2+\beta(t))\}
$$

by making use of (3.13). While, by Lemma 5.9

$$2^{m-s+l-4+2^{l+1}k}\beta_1^d(2+\beta(s-l-1))\beta(s-l)=0 \qquad (2\leqq l\leqq s-2),$$

$$2^{m-s+l-4+2^{l+1}k}\beta_1^d\beta(u)(2+\beta(s-l-1))\beta(s-l)=0 \qquad (2\leqq l\leqq s-3).$$

These imply that

$$(*) = -2^{m-4+2^s k}\beta_1^d\beta(1).$$

Therefore, we have the desired result. q.e.d.

§7. The group $\widetilde{KO}(S^{4n+3}/Q_r)$ ($r=2^{m-1}$) for odd n

In this section, we shall determine the additive structure of $\widetilde{KO}(S^{4n+3}/Q_r)$ ($r = 2^{m-1}$) with $m\geqq 2$ for odd n by giving an additive base. In case $m=1$, $\widetilde{KO}(S^{4n+3}/Q_1) = \widetilde{KO}(L^{2n+1}(4))$ and its additive structure is given in [12, Th. B]. The result in case $m=2$ is given in [7, Th.1.3].

Let $m\geqq 2$. Then, we have the relations in $\widetilde{KO}(S^{4n+3}/Q_r)$ given by the following propositions.

PROPOSITION 7.1. *Suppose $0\leqq s\leqq m-2$ if k is even, $1\leqq s\leqq m-2$ if k is odd, and d is even under the assumption* (6.1). *Then, we have*

$$\textstyle\sum_{t=0}^{s+1}(-1)^{2^t}2^{m-s-4+2^t k}\beta_1^d\beta(s+1-t)+2^{m-s-4+k}R_0(s+1,\ d;\ k)=0,$$

where $R_0(s+1,\ d;\ k)$ is the element

$$(1+(-1)^{2^{k'-1}})2^{m-s-2}\beta_1^d\beta(s+1) \qquad \text{if } k=2k',$$

$$(1+(-1)^{2^{m-s-2}})\beta_1^d\beta(s+1)+(1+(-1)^{2^{s-1}})2^{k'+k}\beta_1^d\beta(s)$$

$$+(1+(-1)^{2^{s-1}})(1+(-1)^{2^{|s-2|}})2^{k'+3k}\beta_1^d\beta(s-1)$$

$$+(1+(-1)^{2^{sk'-1}})(1+(-1)^{2^{|(s-1)k'-1|}})2^{s-1+(2^s-1)k}\beta_1^d\beta(1)$$

$$-(1-(-1)^{2^{sk'-1}})2^{1+3k}\beta_1^{d+1} \qquad \text{if } k=2k'+1.$$

PROOF. Combining Lemmas 6.6, 6.7, 6.11 and 6.12, the desired result follows immediately by making use of Lemma 5.1. q.e.d.

PROPOSITION 7.2. *Suppose $2\leqq s\leqq m-2$ and $d\geqq 2$ is even under the assumption* (6.1). *Then*

$$2^{m-s-3+k}\beta_1^{d-2}\beta(2)\textstyle\prod_{t=1}^{s-1}(2+\beta(t))+2^{m-s-3+k}R(s,\ d;\ k)=0,$$

where $R(s,\ d;\ k)=(-1)^k\sum_{t=0}^{s}2^{-1+(2^{t+1}-1)k}\beta_1^d\beta(s-t)+$

$$(-1)^{(2^{k'-1}+1)\varepsilon(k)+2^{m-s-2}}2^k\beta_1^d\beta(s)+2^{2+k}\beta_1^{d-1}\beta(s)$$

$$+(1-(-1)^{2^{k'-1}})\,\varepsilon(k)2^{1+3k}\beta_1^d\beta(s-1)-2^{s-1+(2^s-1)k}\beta_1^d\beta(1).$$

Here, $k'=[k/2]$ and $\varepsilon(k)=0$ if k is even, $=1$ if k is odd.

PROOF. The desired result follows from Lemmas 6.2, 6.13, 6.14, 6.18, 6.20, 6.27 and 6.28. q.e.d.

PROPOSITION 7.3. *Suppose* $1 \leqq s \leqq m-2$ *and* d *is an odd integer with* $0 < d < 2^s$ *under the assumption* (6.1). *Then, the following relation holds in* $\widetilde{KO}(S^{4n+3}/Q_r)$ *for any non negative integer* n:

$$2^{m-s-2+k}\beta_1^{d-1}\beta(1)\Pi_{t=0}^{s-1}(2+\beta(t)) + \Sigma_{t=0}^{s}(-1)^{2^t}2^{m-s-3+2^{t+1}k}\beta_1^d\beta(s-t) = 0.$$

PROOF. By [8, Lemma 7.3(ii)] and [9, Th.1.7], the relation

$$(*) \quad 2^{m-s-3+k}\beta_1^{d-1}\beta(1)\Pi_{t=0}^{s-1}(2+\beta(t)) + \Sigma_{t=0}^{s}(-1)^{2^t}2^{m-s-4+2^{t+1}k}\beta_1^d\beta(s-t) = 0$$

holds in $\widetilde{K}(S^{4n+3}/Q_r)$. Consider an element $P(2\beta_1, \beta_1^2)$ of $R(Q_r)$ which is a polynomial in $2\beta_1$ and β_1^2. Then $P(2\beta_1, \beta_1^2)$ is an element of $c(RO(Q_r)) \subset R(Q_r)$ by Propositions 2.6 and 2.7. Since $\beta_1 \in R(Q_r)$ is self-conjugate,

$$cr(P(2\beta_1, \beta_1^2)) = (1+t)(P(2\beta_1, \beta_1^2)) = 2P(2\beta_1, \beta_1^2),$$

where $r: R(Q_r) \longrightarrow RO(Q_r)$ is the real restriction and $t: R(Q_r) \longrightarrow R(Q_r)$ is the conjugation. Therefore, we find that the image of $(*)$ by r is the desired relation by making use of the commutative diagram (3.2) and the definitions of β_1, $\beta(t) \in \widetilde{K}(S^{4n+3}/Q_r)$ in [9, (1.1) and (5.1)] and of $2\beta_1$, $\beta(t) \in \widetilde{KO}(S^{4n+3}/Q_r)$ in (3.3) and (3.13), since we identify $RO(Q_r)$ with $c(RO(Q_r))$ under the monomorphic complexification c (cf. §2).

q.e.d.

PROPOSITION 7.4. (i) $2^{n+1}\alpha_0 = 0$ $(m \geqq 2)$.

(ii)
$$2^{n+1}\alpha_1 = \begin{cases} 0 & \text{if } m = 2, \\ \pm 2^{m-1+2n}\beta_1 & \text{if } m \geqq 3. \end{cases}$$

PROOF. (i) By Propositions 2.5 and 2.7,

$$2^{n+1}\alpha_0 = \alpha_0\beta_1^{n+1} \text{ in } \widetilde{RO}(Q_r)$$

and $\alpha_0\beta_1^{n+1} \in \operatorname{Ker} \xi$ by Lemma 3.10. Therefore,

$$2^{n+1}\alpha_0 = 0 \text{ in } \widetilde{KO}(S^{4n+3}/Q_r)$$

by (3.9) and the definitions of α_0, $2\beta_1$ and $\beta_1^2 \in \widetilde{KO}(S^{4n+3}/Q_r)$ in (3.3) (see also Propositions 2.5 and 2.7).

(ii) By Proposition 2.5,

$$\alpha_1\beta_1^{n+1} = \beta_1^n(\beta_{r-1} - \beta_1) - 2\alpha_1\beta_1^n \text{ in } \widetilde{R}(Q_r).$$

On the other hand, by [9, Lemma 5.3]

$$\beta_{r-1} - \beta_1 = \Sigma_{u=1}^{m-2}(2+\beta_1)\beta(u)\Pi_{t=u+1}^{m-2}(2+\beta(t)) \text{ in } \widetilde{R}(Q_r).$$

Thus

$$(*) \qquad \alpha_1\beta_1^{n+1} = (\beta_1^{n+1} + (-1)^n2^{n+1})\Sigma_{u=1}^{m-2}\beta(u)\Pi_{t=u+1}^{m-2}(2+\beta(t)) \text{ in } \widetilde{R}(Q_r).$$

Since the both sides of $(*)$ are the polynomials in α_1 and β_1^2, the same relation as $(*)$ holds in $\widetilde{RO}(Q_r)$ by Proposition 2.7. Also the same relation as $(*)$ holds in $\widetilde{KO}(S^{4n+3}/Q_r)$

by the definitions of α_1, β_1^2 and $\beta(t) \in \widetilde{KO}(S^{4n+3}/Q_r)$ in (3.3) and (3.13). Therefore, we have

$$2^{n+1}\alpha_1 = 2^{n+1}\sum_{u=1}^{m-2}\beta(u)\prod_{t=u+1}^{m-2}(2+\beta(t)) \text{ (by (3.9))}.$$

From this relation, $2^{n+1}\alpha_1 = 0$ if $m=2$. Let $m \geqq 3$. Then, by Lemma 5.1,

$$2^{n+i-1}\beta(m-i) = 0 \quad (2 \leqq i \leqq m-2).$$

Hence, we have

$$2^{n+1}\sum_{u=1}^{m-2}\beta(u)\prod_{t=u+1}^{m-2}(2+\beta(t)) = 2^{n+m-2}\beta(1) = \pm 2^{m-1+2n}\beta_1 \text{ (by Lemmas 6.5 and 5.1)}.$$

q.e.d.

Now, we are ready to prove Theorem 1.6 for odd n.

PROOF OF THEOREM 1.6 **FOR ODD** n. The group $\widetilde{KO}(S^{4n+3}/Q_r)$ $(r=2^{m-1})$ for odd n is additively generated by α_0, $\bar{\alpha}_1$ and $\bar{\delta}_i$ $(1 \leqq i \leqq N')$ by Propositions 2.5, 2.7, (3.9), (3.13) and the fact that $2P_{m,1} = \beta_1 P_{m,1} = 0$ in Lemma 3.14(ii). On the other hand, $2^{n+1} \times 2^{n+1} \times \prod_{i=1}^{N'}\bar{u}(i) = 2^{(m+3)n+2} = \#\widetilde{KO}(S^{4n+3}/Q_r)$ by Propositions 4.13(ii), 7.1–4, Lemma 5.1 and the definitions of $\bar{\alpha}_1$, $\bar{u}(i)$ and $\bar{\delta}_i$ $(1 \leqq i \leqq N')$ in (1.5). Therefore, we complete the proof of Theorem 1.6 for odd n. q.e.d.

COROLLARY 7.5 (cf. [13, Cor.1.7]). *The order of* $\bar{\delta}_1$ *in* $\widetilde{KO}(S^{4n+3}/Q_r)$ *is equal to* 2^{m+2n-1} *if* n *is an odd integer.*

§8. Some relations in $\widetilde{KO}(S^{4n+3}/Q_r)$ $(r=2^{m-1})$ for even n

In this section, we give some relations in $\widetilde{KO}(S^{4n+3}/Q_r)$ $(r=2^{m-1} \geqq 2)$ for even n, which play an important part in the next section.

For the elements $2\beta(0)$, $\beta(s) \in \widetilde{KO}(S^{4n+3}/Q_r)$ $(1 \leqq s \leqq m)$ in (3.13), we have the following lemmas.

LEMMA 8.1. *For any integers* $k_0, \cdots, k_{s-1} \geqq 0$ *and* $k_s > 0$ $(0 \leqq s \leqq m)$, *we have the following relations*:

$$(1)_s \begin{cases} 2^{m-s+h}\prod_{t=0}^{s}\beta(t)^{k_t} = 0 & \text{if } s=0,\ 1 \text{ and } m-s+h>0, \\ 2^{m-s+h+\varepsilon(k_0)}\prod_{t=0}^{s}\beta(t)^{k_t} = 0 & \text{if } 2 \leqq s \leqq m \text{ and } m-s+h>0, \end{cases}$$

$$(2)_s \quad 2^{\varepsilon(k_0)}\prod_{t=0}^{s}\beta(t)^{k_t} = 0 \quad \text{if } m-s+h \leqq 0,$$

where $h = h(k_0, \cdots, k_s) = 1 + [(n - \sum_{t=0}^{s}2^t k_t)/2^{s-1}]$ *and* $\varepsilon(k_0) = 0$ *if* k_0 *is even*, $= 1$ *if* k_0 *is odd.*

PROOF. We prove the lemma by the induction on s and h. Let $s=0$, and suppose that $h(k_0) < 0$. Then $k_0 \geqq n+1$ and $2\beta_1^{n+1} = 0 = \beta_1^{n+2}$ by (3.9) and Lemma 3.10. Thus $(1)_0$ and $(2)_0$ for $h(k_0) < 0$ hold. Suppose that $h = h(k_0) \geqq 0$, and assume that $(1)_0$ and $(2)_0$ hold for any k_0 with $h(k_0) < h$. Since $h = h(k_0) = 1 + 2(n-k_0) > 0$ and n is even,

$$2^{h-1}\beta_1^{k_0-1}P_{m,1}=0$$

by Lemma 3.14, and so

$$(*)\qquad 2^{m+h}\beta(0)^{k_0}+2^{m-2+h}\beta(0)^{k_0+1}+\textstyle\sum_{I_0}2^{m-2+h-j}\beta(0)^{k_0-1}\beta(1)\beta(i_1)\cdots\beta(i_j)=0,$$

where $I_0=\{(i_1,\cdots,i_j): 1\leqq j\leqq m-1,\ 0\leqq i_1<\cdots<i_j\leqq m-2\}$.
By making use of (3.13) and the inductive hypothesis, the second term and the term for any $(i_1,\cdots,i_j)\in I_0$ in $(*)$ vanish. Thus, $(1)_0$ and $(2)_0$ hold.

Let $s=1$, and suppose that $h=h(k_0,k_1)<0$. By (3.13),

$$2^{m-1+h}\beta(0)^{k_0}\beta(1)^{k_1}=\textstyle\sum_{i=0}^{k_1}\binom{k_1}{i}2^{m-1+h+2i}\beta(0)^{k_0+2k_1-i}\quad \text{if } m-1+h>0,$$

$$2^{\varepsilon(k_0)}\beta(0)^{k_0}\beta(1)^{k_1}=\textstyle\sum_{i=0}^{k_1}\binom{k_1}{i}2^{\varepsilon(k_0)+2i}\beta(0)^{k_0+2k_1-i}.$$

If $m-1+h>0$,

$$2^{m-1+h+2i}\beta(0)^{k_0+2k_1-i}=0\quad(0\leqq i\leqq k_1)$$

by $(1)_0$ and $(2)_0$. Thus $(1)_1$ for $h<0$ holds. If $m-1+h\leqq 0$,

$$2^{\varepsilon(k_0)+2i}\beta(0)^{k_0+2k_1-i}=0\quad(0\leqq i\leqq k_1)$$

by $(1)_0$ and $(2)_0$. Hence $(2)_1$ for $h<0$ holds. Suppose $h=h(k_0,k_1)\geqq 0$, and assume that $(1)_1$ and $(2)_1$ hold for any k_0, k_1 with $h(k_0,k_1)<h$. Since $h=1+n-k_0-2k_1\geqq 0$ and n is even,

$$2^{h}\beta(0)^{k_0}\beta(1)^{k_1-1}P_{m,1}=0$$

by Lemma 3.14, and so

$$(**)\quad 2^{m-1+h}\beta(0)^{k_0}\beta(1)^{k_1}+2^{m-2+h}\beta(0)^{k_0+1}\beta(1)^{k_1}$$
$$+\textstyle\sum_{I_1}2^{m-2+h-j}(2+\beta(0))\beta(0)^{k_0}\beta(1)^{k_1}\beta(i_1)\cdots\beta(i_j)=0,$$

where $I_1=\{(i_1,\cdots,i_j): 1\leqq j\leqq m-2,\ 1\leqq i_1<\cdots<i_j\leqq m-2\}$.
By the inductive hypothesis and (3.13), the second term and the term for any $(i_1,\cdots,i_j)\in I_1$ in $(**)$ vanish. Thus, $(1)_1$ and $(2)_1$ hold.

Let $2\leqq s\leqq m$. Suppose $h=h(k_0,\cdots,k_s)<0$, and assume that $(1)_{s-1}$ and $(2)_{s-1}$ hold. Then, by (3.13),

$$2^{m-s+h+\varepsilon(k_0)}\alpha\beta(s)^{k_s}=\textstyle\sum_{i=0}^{k_s}\binom{k_s}{i}2^{m-s+h+\varepsilon(k_0)+2i}\alpha\beta(s-1)^{2k_s-i},$$

$$2^{\varepsilon(k_0)}\alpha\beta(s)^{k_0}=\textstyle\sum_{i=0}^{k_s}\binom{k_s}{i}2^{\varepsilon(k_0)+2i}\alpha\beta(s-1)^{2k_s-i},$$

where $\alpha=\prod_{t=0}^{s-1}\beta(t)^{k_t}$. If $m-s+h>0$,

$$2^{m-s+h+\varepsilon(k_0)+2i}\alpha\beta(s-1)^{2k_s-i}=0\quad(0\leqq i\leqq k_s)$$

by $(1)_{s-1}$ and $(2)_{s-1}$, and so $(1)_s$ for $h<0$ holds. If $m-s+h\leqq 0$,

$$2^{\varepsilon(k_0)+2i}\alpha\beta(s-1)^{2k_s-i}=0\quad(0\leqq i\leqq k_s)$$

by $(1)_{s-1}$ and $(2)_{s-1}$, and so $(2)_s$ for $h<0$ holds. Suppose $h=h(k_0,\cdots,k_s)\geqq 0$, and assume that $(1)_s$ and $(2)_s$ hold for any $k_0,\cdots,k_s$ with $h(k_0,\cdots,k_s)<h$. By Lemma 3.14,

$$2^{h+\varepsilon(k_0)}\alpha\beta(s)^{k_s-1}P_{m,s}=0 \quad (\alpha=\Pi_{t=0}^{s-1}\beta(t)^{k_t}),$$

and so

$$(***)\quad 2^{m-s+h+\varepsilon(k_0)}\alpha\beta(s)^{k_s}+2^{m-s-1+h+\varepsilon(k_0)}\alpha\beta(s-1)\beta(s)^{k_s}$$
$$+\sum_{I_s}2^{m-s-1-j+h+\varepsilon(k_0)}(2+\beta(s-1))\alpha\beta(s)^{k_s}\beta(i_1)\cdots\beta(i_j)=0,$$

where $I_s=\{(i_1,\cdots,i_j):1\leqq j\leqq m-1-s,\ s\leqq i_1<\cdots<i_j\leqq m-2\}$.

By the inductive hypothesis and (3.13), the second term and the term for any $(i_1,\cdots,i_j)\in I_s$ in $(***)$ vanish. Therefore, $(1)_s$ and $(2)_s$ for $h\geqq 0$ hold. q.e.d.

We can prove the following lemma in the similar way to the proof of Lemma 5.2 by making use of Lemma 8.1 and (3.13).

LEMMA 8.2. *For any integers $k_0,\cdots,k_{s-1}\geqq 0$ and $k_s>l>0$ $(0\leqq s\leqq m)$, we have*

$$2^{m-s+\varepsilon+h'}\alpha\beta(s)^{k_s}=(-1)^l2^{m-s+\varepsilon+h'+2l}\alpha\beta(s)^{k_s-1}\quad \textit{if } m-s+h'>0.$$

Also

$$2^{\varepsilon(k_0)}\alpha\beta(s)^{k_s}=-2^{\varepsilon(k_0)+2}\alpha\beta(s)^{k_s-1}\quad \textit{if } k_s\geqq 2 \textit{ and } m-s+h'\leqq 0.$$

Here, $h'=[(n-\Pi_{t=0}^{s}2^tk_t)/2^s]$, $\alpha=\Pi_{t=0}^{s-1}\beta(t)^{k_t}$ *and* $\varepsilon=0$ *if* $s=0$, $=\varepsilon(k_0)$ *if* $1\leqq s\leqq m$.

The following lemma is obtained in the similar way to the proof of Lemma 5.3 by making use of Lemma 8.1 and (3.13).

LEMMA 8.3. *Let $h=h(k_0,\cdots,k_s)$ be the one in Lemma 8.1 and $\alpha=\Pi_{t=0}^{s-1}\beta(t)^{k_t}$. Then we have*

$$\text{(i)}\begin{cases}2^{m-1+2h}\beta(0)^{k_0}\beta(1)^{k_1}=0 & \textit{if } s=1,\ m-2+2h\geqq 0,\\ 2^{\varepsilon(k_0)}\beta(0)^{k_0}\beta(1)^{k_1}=0 & \textit{if } s=1,\ m-2+2h<0.\end{cases}$$

$$\text{(ii)}\begin{cases}2^{m-s+1+2h+\varepsilon(k_0)}\alpha\beta(s)^{k_s}=0 & \textit{if } 2\leqq s\leqq m,\ m-s+1+2h\geqq 0,\\ 2^{\varepsilon(k_0)}\alpha\beta(s)^{k_s}=0 & \textit{if } 2\leqq s\leqq m,\ m-s+1+2h<0.\end{cases}$$

LEMMA 8.4. *Let $m\geqq 3$, $l\geqq 1$ and $l\geqq h=h(k_0,k_1)$ except for the case $l=1$ and h is even. Then*

$(1)_h$ $\pm(2+\beta(0))2^{m-4+l}\beta(0)^{k_0+1}\beta(1)^{k_1}=(2+\beta(0))2^{m-3+l}\beta(0)^{k_0+1}\beta(1)^{k_1-1}$ *if* $k_0\geqq 0$ *and* $k_1\geqq 2$,

$(2)_h$ $\pm(2+\beta(0))2^{m-4+l}\beta(0)^{k_0-1}\beta(1)^{k_1+1}=(2+\beta(0))2^{m-3+l}\beta(0)^{k_0-1}\beta(1)^{k_1}$ *if* $k_0>0$, $k_1>0$.

PROOF. By Lemma 3.14,

$$2^{l-1}\beta(0)^{k_0+1}\beta(1)^{k_1-2}P_{m,1}=0,$$

and so

$$2^{m-3+l}(2+\beta(0))\beta(0)^{k_0+1}\beta(1)^{k_1-1}+\Sigma_{I_1}2^{m-3+l-j}(2+\beta(0))\beta(0)^{k_0+1}\beta(1)^{k_1-1}\beta(i_1)\cdots\beta(i_j)=0.$$

The terms for $(i_1, \cdots, i_j)\in I_1$ vanish except for $(1)\in I_1$ by Lemma 8.1. This implies $(1)_h$. $(2)_h$ follows from the relation

$$2^{l-1}\beta(0)^{k_0-1}\beta(1)^{k_1-1}P_{m,1}=0$$

in Lemma 3.14 by making use of Lemma 8.1 in the similar way to the proof of $(1)_h$.

q.e.d.

Lemma 8.5. *Let* $m\geqq 3$, $l\geqq 1$ *and* $l\geqq h=h(k_0, k_1)$. *Then*

$(3)_h$ $\quad 2^{m-1+l}\beta(0)^{k_0+1}\beta(1)^{k_1-1}\pm 2^{m-2+l}\beta(0)^{k_0}\beta(1)^{k_1}=0$ *if* $k_0\geqq 0$ *and* $k_1\geqq 2$,

$(4)_h$ $\quad 2^{m-1+l}\beta(0)^{k_0-1}\beta(1)^{k_1}\pm 2^{m-2+l}\beta(0)^{k_0}\beta(1)^{k_1}=0$ *if* $k_0>0$ *and* $k_1>0$,

$(5)_h$ $\quad 2^{m-2+l}\beta(0)^{k_0}\beta(1)^{k_1}=\pm 2^{m+l}\beta(0)^{k_0}\beta(1)^{k_1-1}$ *if* $k_0\geqq 0$ *and* $k_1\geqq 2$.

Proof. If $l=1\geqq h$ and h is even, each term in $(3)_h$, $(4)_h$ and $(5)_h$ vanishes by Lemma 8.1. In other cases, $(3)_h$ and $(4)_h$ follow from $2\times(1)_h$ and $2\times(2)_h$ in Lemma 8.4 by (3.13) and Lemma 8.1. $(5)_h$ is the immediate consequence of $(3)_h$ and $(4)_h$.

q.e.d.

Lemma 8.6. *Let* $m\geqq 3$ *and* $h(k_0, k_1)=1$. *Then*

(6) $\quad -2^{m-1}\beta(0)^{k_0+1}\beta(1)^{k_1-1}+2^{m-2}\beta(0)^{k_0}\beta(1)^{k_1}\pm 2^{m-2+2n}\beta(0)=0$ *if* $k_0\geqq 0$ *and* $k_1\geqq 2$,

(7) $\quad 2^{m-1}\beta(0)^{k_0-1}\beta(1)^{k_1}+2^{m-2}\beta(0)^{k_0}\beta(1)^{k_1}\pm 2^{m-2+2n}\beta(0)=0$ *if* $k_0>0$ *and* $k_1>0$.

Proof. Consider $(1)_1$ for $l=1=h(k_0, k_1)$ in Lemma 8.4. The term

$$2^{m-3}\beta(0)^{k_0+2}\beta(1)^{k_1}$$

vanishes by Lemma 8.3. By (3.13),

$(*)$ $\quad 2^{m-2}\beta(0)^{k_0+1}\beta(1)^{k_1}=\Sigma_{i=0}^{k_1}\binom{k_1}{i}2^{m-2+2i}\beta(0)^{k_0+1+2k_1-i}$

The term for $i=0$ in $(*)$ vanishes by Lemma 8.1, and the term for $i\geqq 1$ in $(*)$ is equal to

$$\binom{k_1}{i}2^{m-2+2i}\beta(0)^{k_0+1+2k_1-i}=\pm\binom{k_1}{i}2^{m-2+2k_0+2^2k_1}\beta(0)$$

by Lemmas 8.1–2. Therefore, we have

$$2^{m-2}\beta(0)^{k_0+1}\beta(1)^{k_1}=\pm 2^{m-2+2n}\beta(0).$$

On the other hand, by (3.13)

$$2^{m-2}\beta(0)^{k_0+2}\beta(1)^{k_1-1}=2^{m-2}\beta(0)^{k_0}\beta(1)^{k_1}-2^{m}\beta(0)^{k_0+1}\beta(1)^{k_1-1}.$$

Thus (6) follows from Lemma 8.4$(1)_1$. (7) follows from Lemma 8.4$(2)_1$ in the similar way to the proof of (6).

q.e.d.

LEMMA 8.7. *Let* $m \geqq 3$ *and* $h(k_0, k_1) = 2$. *Then*

(8) $2^m \beta(0)^{k_0+1}\beta(1)^{k_1-1} + 2^{m-1}\beta(0)^{k_0}\beta(1)^{k_1} \pm 2^{m-2+2n}\beta(0) = 0$ *if* $k_0 \geqq 0$ *and* $k_1 \geqq 2$,

(9) $-2^m \beta(0)^{k_0-1}\beta(1)^{k_1} + 2^{m-1}\beta(0)^{k_0}\beta(1)^{k_1} \pm 2^{m-2+2n}\beta(0) = 0$ *if* $k_0 > 0$ *and* $k_1 > 0$.

PROOF. Consider $(1)_2$ for $l = 2 = h(k_0, k_1)$ in Lemma 8.4.

Then

$$2^{m-1}\beta(0)^{k_0+1}\beta(1)^{k_1} = \pm 2^{m+1}\beta(0)^{k_0+1}\beta(1)^{k_1-1} \text{ (by Lemma 8.5(5)}_1\text{)}$$

By (3.13), we have

$$(*) \quad 2^{m-2}\beta(0)^{k_0+2}\beta(1)^{k_1} = \textstyle\sum_{i=0}^{k_1}\binom{k_1}{i} 2^{m-2+2i}\beta(0)^{k_0+2k_1+2-i}.$$

The term for $i = 0$ in $(*)$ vanishes by Lemma 8.1, and

$$2^{m-2+2i}\beta(0)^{k_0+2k_1+2-i} = \pm 2^{m-2+2n}\beta(0) \text{ if } i \geqq 1$$

by Lemmas 8.1–2. On the other hand, by (3.13) and Lemma 8.1,

$$2^{m-1}\beta(0)^{k_0+2}\beta(1)^{k_1-1} = 2^{m-1}\beta(0)^{k_0}\beta(1)^{k_1} \pm 2^{m+1}\beta(0)^{k_0+1}\beta(1)^{k_1-1}.$$

Therefore, (8) follows from Lemma 8.4$(1)_2$. (9) follows from Lemmas 8.4$(2)_2$ and 8.5$(5)_1$ in the similar way to the proof of (8). q.e.d.

LEMMA 8.8. *Let* $m \geqq 3$, $l \geqq 3$ *and* $l \geqq h = h(k_0, k_1)$. *Then*

$(10)_h$ $2^{m-2+l}\beta(0)^{k_0+1}\beta(1)^{k_1-1} - 2^{m-3+l}\beta(0)^{k_0}\beta(1)^{k_1} = 0$ *if* $k_0 \geqq 0$, $k_1 \geqq 2$,

$(11)_h$ $2^{m-2+l}\beta(0)^{k_0-1}\beta(1)^{k_1} + 2^{m-3+l}\beta(0)^{k_0}\beta(1)^{k_1} = 0$ *if* $k_0 > 0$, $k_1 > 0$.

PROOF. Consider $(1)_h$ in Lemma 8.4. Then

$$(2+\beta(0))2^{m-4+l}\beta(0)^{k_0+1}\beta(1)^{k_1} = 0$$

by Lemma 8.5$(4)_{h-2}$. Also, by (3.13),

$$2^{m-3+l}\beta(0)^{k_0+2}\beta(1)^{k_1-1} = 2^{m-3+l}\beta(0)^{k_0}\beta(1)^{k_1} - 2^{m-1+l}\beta(0)^{k_0+1}\beta(1)^{k_1-1}.$$

Therefore, $(10)_h$ follows from Lemma 8.4$(1)_h$. $(11)_h$ follows from Lemmas 8.4$(2)_h$ and 8.5$(4)_{h-2}$ in the similar way to the proof of $(10)_h$. q.e.d.

LEMMA 8.9. *Let* $2 \leqq s \leqq m-1$, $l \geqq 1$ *and* $l \geqq h = h(k_0, \cdots, k_s)$.

Then

$(12)_h$ $(2+\beta(s-1))2^{m-s-2+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1}$

$$= \begin{cases} \pm(2+\beta(s-1))2^{m-s-3+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s} & \text{if } 2 \leqq s \leqq m-2,\ k_{s-1} \geqq 0 \text{ and } k_s \geqq 2, \\ 0 & \text{if } s = m-1,\ k_{s-1} \geqq 0 \text{ and } k_s \geqq 2, \end{cases}$$

$(13)_h$ $(2+\beta(s-1))2^{m-s-2+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}-1}\beta(s)^{k_s}$

$$= \begin{cases} \pm(2+\beta(s-1))2^{m-s-3+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}-1}\beta(s)^{k_s+1} & \text{if } 2 \leqq s \leqq m-2,\ k_{s-1}>0 \text{ and } k_s>0, \\ 0 & \text{if } s=m-1,\ k_{s-1}>0 \text{ and } k_s>0, \end{cases}$$

where $\alpha=\prod_{t=0}^{s-2}\beta(t)^{k_t}$. *Moreover, the right hand sides of* $(12)_h$ *and* $(13)_h$ *vanish if* $h \leqq 0$.

PROOF. By Lemma 3.14,

$$2^{l-1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-2}P_{m,s}=0,$$

and so

$$(2+\beta(s-1))2^{m-s-2+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1}+$$
$$\sum_{I_s}(2+\beta(s-1))2^{m-s-2+l-j+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1}\beta(i_1)\cdots\beta(i_j)=0.$$

Since $I_{m-1}=\phi$, $(12)_h$ for $s=m-1$ holds. Consider the case $2 \leqq s \leqq m-2$. Then, the terms for $(i_1, \cdots, i_j) \in I_s$ vanish except for $(s) \in I_s$ by Lemma 8.1. Therefore, $(12)_h$ holds. $(13)_h$ follows from the relation

$$2^{l-1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}-1}\beta(s)^{k_s-1}P_{m,s}=0$$

in Lemma 3.14 in the same manner as the proof of $(12)_h$. The last statement is easily verified by Lemma 8.1. q.e.d.

LEMMA 8.10. *Let* $2 \leqq s \leqq m-1$, $l \geqq 0$ and $l \geqq h=h(k_0, \cdots, k_s)$. *Then*

$(14)_h$ $2^{m-s+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1}$

$= \pm 2^{m-s-1+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s}$ *if* $k_{s-1} \geqq 0$, $k_s \geqq 2$,

$(15)_h$ $2^{m-s+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}-1}\beta(s)^{k_s} = \pm 2^{m-s-1+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s}$ *if* $k_{s-1}>0$, $k_s>0$,

$(16)_h$ $2^{m-s-1+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s} = \pm 2^{m-s+1+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s-1}$ *if* $k_{s-1} \geqq 0$, $k_s \geqq 2$,

where $\alpha=\prod_{t=0}^{s-2}\beta(t)^{k_t}$.

PROOF. Since

$$2^{m-s+1+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1}=0$$

by Lemma 8.1, $(14)_h$ and $(15)_h$ follow from $(12)_h$ and $(13)_h$ respectively by making use of (3.13) and Lemma 8.1. $(16)_h$ is the immediate consequence of $(14)_h$ and $(15)_h$. q.e.d.

LEMMA 8.11. *Let* $2 \leqq s \leqq m-2$ *and* $h(k_0, \cdots, k_s)=1$. *Then*

(17) $2^{m-s+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1}+2^{m-s-1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s}$

$= \pm 2^{m-s+2k_{s-1}+2^2k_s+\varepsilon(k_0)}\alpha\beta(s-1)$ *if* $k_{s-1} \geqq 0$, $k_s \geqq 2$,

(18) $2^{m-s+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}-1}\beta(s)^{k_s}-2^{m-s-1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s}$

$= \pm 2^{m-s+2k_{s-1}+2^2k_s+\varepsilon(k_0)}\alpha\beta(s-1)$ *if* $k_{s-1}>0$, $k_s>0$,

where $\alpha=\prod_{t=0}^{s-2}\beta(t)^{k_t}$. *Moreover, the right hand sides of* (17) *and* (18) *vanish if* $s=2$ *or* $0 \leqq n-\sum_{t=0}^{s}2^t k_t < 2^{s-2}$.

PROOF. Consider $(12)_1$ in Lemma 8.9. By (3.13)

$$2^{m-s-2+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+2}\beta(s)^{k_s}=\sum_{i=0}^{k_s}\binom{k_s}{i}2^{m-s-2+\varepsilon(k_0)+2i}\alpha\beta(s-1)^{k_{s-1}+2k_s+2-i}.$$

The term for $i=0$ vanishes by Lemma 8.3, and

$$2^{m-s-2+\varepsilon(k_0)+2i}\alpha\beta(s-1)^{k_{s-1}+2k_s+2-i}=\pm 2^{m-s+2k_{s-1}+2^2k_s+\varepsilon(k_0)}\alpha\beta(s-1)\ \text{ if }\ i\geqq 1$$

by Lemma 8.2. Therefore, we have

$$2^{m-s-2+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+2}\beta(s)^{k_s}=\pm 2^{m-s+2k_{s-1}+2^2k_s+\varepsilon(k_0)}\alpha\beta(s-1).$$

On the other hand, by $(16)_0$ in Lemma 8.10

$$2^{m-s-1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s}=\pm 2^{m-s+1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1}.$$

Hence, we have (17) by $(12)_1$ in Lemma 8.9. In the same manner as the proof of (17), we have (18) by making use of $(13)_1$ in Lemma 8.9, $(16)_0$ in Lemma 8.10 and Lemma 8.1. The last statement follows from Lemma 8.1. q.e.d.

LEMMA 8.12. *Let* $2\leqq s\leqq m-2$, $l\geqq 2$ *and* $l\geqq h=h(k_0,\cdots,k_s)$. *Then*

$(19)_h$ $2^{m-s-1+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1}$

$$=2^{m-s-2+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s}\qquad \textit{if } k_{s-1}\geqq 0,\ k_s\geqq 2,$$

$(20)_h$ $2^{m-s-1+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}-1}\beta(s)^{k_s}$

$$=-2^{m-s-2+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s}\quad \textit{if } k_{s-1}>0,\ k_s>0.$$

PROOF. Consider the right hand side of $(12)_h$ in Lemma 8.9. Then, we have

$$2^{m-s-2+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s}=\pm 2^{m-s+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1}$$

by $(16)_{h-1}$ in Lemma 8.10, and

$$2^{m-s-3+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+2}\beta(s)^{k_s}=\pm 2^{m-s-1+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+2}\beta(s)^{k_s-1}$$

by $(16)_{h-2}$ in Lemma 8.10. On the other hand,

$$2^{m-s-2+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+2}\beta(s)^{k_s-1}$$

$$=2^{m-s-2+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s}-2^{m-s+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}+1}\beta(s)^{k_s-1}$$

by (3.13). Thus, $(19)_h$ follows from $(12)_h$. Consider $(13)_h$ in Lemma 8.9. Then, we have

$$2^{m-s-2+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}-1}\beta(s)^{k_s+1}=\pm 2^{m-s+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}-1}\beta(s)^{k_s}$$

by $(16)_{h-1}$, and

$$2^{m-s-3+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s+1}=\pm 2^{m-s-1+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s}$$

by $(16)_{h-2}$. Thus $(20)_h$ follows from $(13)_h$. q.e.d.

The following lemma is obtained from Lemmas 8.6–12.

LEMMA 8.13. (i) *Let* $m \geqq 3$, $k_0 \geqq 0$ *and* $k_1 \geqq 2$. *Then*

(21) $2^{m-2}\beta(0)^{k_0}\beta(1)^{k_1} = 2^m\beta(0)^{k_0}\beta(1)^{k_1-1}$ *if* $h(k_0, k_1) = 1$,

(22) $2^{m-1}\beta(0)^{k_0}\beta(1)^{k_1} = 2^{m+1}\beta(0)^{k_0}\beta(1)^{k_1-1} \pm 2^{m-2+2n}\beta(0)$ *if* $h(k_0, k_1) = 2$,

$(23)_h$ $2^{m-3+l}\beta(0)^{k_0}\beta(1)^{k_1} = -2^{m-1+l}\beta(0)^{k_0}\beta(1)^{k_1-1}$ *if* $l \geqq 3$ *and* $l \geqq h(k_0, k_1)$.

(ii) *Let* $2 \leqq s \leqq m-2$, $k_{s-1} \geqq 0$ *and* $k_s \geqq 2$. *Then*

(24) $2^{m-s-1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s} = \pm 2^{m-s+1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s-1}$ *if* $0 \geqq h(k_0, \cdots, k_s)$,

(25) $2^{m-s-1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s}$

$= 2^{m-s+1+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s-1} \pm 2^{m-s+2k_{s-1}+2^2k_s+\varepsilon(k_0)}\alpha\beta(s-1)$ *if* $h(k_0, \cdots, k_s) = 1$,

$(26)_h$ $2^{m-s-2+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s}$

$= -2^{m-s+l+\varepsilon(k_0)}\alpha\beta(s-1)^{k_{s-1}}\beta(s)^{k_s-1}$ *if* $l \geqq 2$ *and* $l \geqq h(k_0, \cdots, k_s)$,

where $\alpha = \prod_{t=0}^{s-2}\beta(t)^{k_t}$. *Moreover, the last term of* (25) *vanishes if* $s = 2$ *or* $0 \leqq n - \sum_{t=0}^{s} 2^t k_t < 2^{s-2}$.

LEMMA 8.14. *Let* $m \geqq 3$ *and* $1 \leqq h \leqq n-2$. *Then*

$$2^{m-2+h}\beta(0)^{n+1-h} = (-1)^{h+1}\{(2^{n-h} - 1)2^{m-3+n}\beta(0)^2 + (2^{n-h-1} - 1)2^{m-1+n}\beta(0)\}.$$

PROOF. Consider $(4)_h$ for $l = h = h(k_0, k_1)$ and $k_0 > 0$, $k_1 = 1$. Then

$$(*) \qquad 2^{m-2+h}\beta(0)^{n+1-h} + 3\ 2^{m-1+h}\beta(0)^{n-h} + 2^{m+1+h}\beta(0)^{n-1-h} = 0$$

by (3.13), where we notice that

$$1 \leqq h = h(k_0, 1) = n - k_0 - 1 \leqq n - 2.$$

The desired result is obtained by the induction on h by making use of $(*)$. q.e.d.

§9. Basic relations concerned with an additive base of $\widetilde{KO}(S^{4n+3}/Q_r)$ $(r = 2^{m-1})$ for even n

In this section, we prove some basic relations concerned with an additive base of $\widetilde{KO}(S^{4n+3}/Q_r)$ $(r = 2^{m-1})$ for even n by making use of the relations given in §8.

Let s, k and d be the integers which satisfy

$$0 \leqq s \leqq m-2,\ 2^s(k-1) \leqq n-d < 2^s k,\ k \geqq 2 \text{ and } d \geqq 0 \text{ (cf. (6.1))}.$$

Then, we have the following lemmas.

LEMMA 9.1. *Suppose* $1 \leqq s \leqq m-2$, k *and* d *are even under the assumption* (6.1). *Then*

$$2^{m-s-2}\beta_1^d(\beta(s+2-t)^{2^{t-2}k} - \beta(s+1-t)^{2^{t-1}k}) = 2^{m-s-4+2^t k}\beta_1^d\beta(s+1-t)$$

for any t *with* $1 \leqq t \leqq s+1$.

PROOF. Let $u=s+1-t$. Then, by (3.13)

$$2^{m-s-2}\beta_1^d(\beta(u+1)^{2^{t-2}k}-\beta(u)^{2^{t-1}k}) = \sum_{i=1}^{2^{t-2}k}\binom{2^{t-2}k}{i}2^{m-s-2+2i}\beta_1^d\beta(u)^{2^{t-1}k-i}.$$

The i-th term is equal to

$$(-1)^{i-1}\binom{2^{t-2}k}{i}2^{m-s-4+2^t k}\beta_1^d\beta(u) \quad (1 \leqq i \leqq 2^{t-2}k)$$

by Lemma 8.2. Therefore, we have the desired result. q.e.d.

LEMMA 9.2. *Under the same assumption as in Lemma* 9.1, *we have*

$$\sum_{t=0}^{s+1}2^{m-s-4+2^t k}\beta_1^d\beta(s+1-t)=0.$$

PROOF. By summarizing the relations of Lemma 9.1 over t, we have

$$\sum_{t=1}^{s+1}2^{m-s-4+2^t k}\beta_1^d\beta(s+1-t)=2^{m-s-2}\beta_1^d\beta(s+1)^{k/2}-2^{m-s-2}\beta_1^{d+2^s k}.$$

By Lemma 8.1, $2^{m-s-2}\beta_1^{d+2^s k}=0$, and

$$2^{m-s-2}\beta_1^d\beta(s+1)^{k/2}=\pm 2^{m-s-4+k}\beta_1^d\beta(s+1)$$

by Lemmas 8.1–2. Therefore, we have the desired result. q.e.d.

LEMMA 9.3. *Suppose* $1 \leqq s \leqq m-2$, $k=2k'+1 \geqq 3$ *and* d *is even under the assumption* (6.1). *Then*

$$2^{m-s-2}\beta_1^d(\beta(s+2-t)^{2^{t-2}k}-\beta(s+1-t)^{2^{t-1}k})$$

$$=\begin{cases} 2^{m-s-4+2^2 k}\beta_1^d\beta(s-1)\pm 2^{m-s-3+2k}\beta_1^d\beta(s) & \text{if } 2=t\leqq s+1, \\ 2^{m-s-4+2^t k}\beta_1^d\beta(s+1-t) & \text{if } 3\leqq t\leqq s+1. \end{cases}$$

PROOF. Let $2 \leqq t \leqq s+1$ and $u=s+1-t$. Then, by (3.13)

$$2^{m-s-2}\beta_1^d\beta(u+1)^{2^{t-2}k}=2^{m-s-2}\beta_1^d(\beta(u)^2+2^2\beta(u))^{2^{t-1}k'}\beta(u+1)^{2^{t-2}}$$

$$=\sum_{i=0}^{2^{t-1}k'}\binom{2^{t-1}k'}{i}2^{m-s-2+2i}\beta_1^d\beta(u)^{2^t k'-i}\beta(u+1)^{2^{t-2}}.$$

Since the i-th term for $1 \leqq i \leqq 2^{t-1}k'$ vanishes by Lemma 8.1,

$$2^{m-s-2}\beta_1^d\beta(u+1)^{2^{t-2}k}=2^{m-s-2}\beta_1^d\beta(u)^{2^t k'}\beta(u+1)^{2^{t-2}}.$$

Thus,

$$(*) \quad 2^{m-s-2}\beta_1^d(\beta(u+1)^{2^{t-2}k}-\beta(u)^{2^{t-1}k})=\sum_{i=1}^{2^{t-2}}\binom{2^{t-2}}{i}2^{m-s-2+2i}\beta_1^d\beta(u)^{2^{t-1}k-i}.$$

The i-th term for $i \neq 1,\ 2\ (3 \leqq t \leqq s+1)$ in $(*)$ is equal to

$$(-1)^{i-1}\binom{2^{t-2}}{i}2^{m-s-4+2^t k}\beta_1^d\beta(u) \quad \text{(by Lemma 8.2)}.$$

The i-th term for $i=2^v (v=0,\ 1$, and $v=0$ if $t=2)$ in $(*)$ is equal to

$$\binom{2^{t-2}}{i}2^{m-s-2+2i}\beta_1^d\beta(u)^{2-i}(\beta(u+1)-2^2\beta(u))^{2^{t-2}k-1} \quad \text{(by (3.13))}$$

$$=\binom{2^{t-2}}{i}\{2^{m-s-2+2i}\beta_1^d\beta(u)^{2-i}\beta(u+1)^{2^{t-2}k-1}+$$
$$\sum_{j=1}^{2^{t-2}k-1}(-1)^j\binom{2^{t-2}k-1}{j}2^{m-s-2+2i+2j}\beta_1^d\beta(u)^{2-i+j}\beta(u+1)^{2^{t-2}k-1-j}\}$$
$$=\pm 2^{m-u-3-v+2i}\beta_1^d\beta(u)^{2-i}\beta(u+1)^{2^{t-2}k-1}+$$
$$(-1)^{2^{t-2}k-1}\binom{2^{t-2}}{i}2^{m-s-4+2i+2^{t-1}k}\beta_1^d\beta(u)^{2^{t-2}k+1-i}\ \text{(by Lemma 8.1)}$$
$$=\pm 2^{m-u-7-v+2i+2^{t-1}k}\beta_1^d\beta(u)^{2-i}\beta(u+1)+(-1)^{i-1}\binom{2^{t-2}}{i}2^{m-s-4+2^tk}\beta_1^d\beta(u)\ \text{(by Lemma 8.2).}$$

Therefore, we have

$$2^{m-s-2}\beta_1^d(\beta(u+1)^{2^{t-2}k}-\beta(u)^{2^{t-1}k})$$
$$=\begin{cases}2^{m-s-4+2^2k}\beta_1^d\beta(u)\pm 2^{m-s-4+2k}\beta_1^d\beta(u)\beta(u+1) & \text{if } 2=t\leqq s+1,\\[2ex] 2^{m-s-4+2^tk}\beta_1^d\beta(u)\pm 2^{m-u-5+2^{t-1}k}\beta_1^d(2+\beta(u))\beta(u+1) & \text{if } 3\leqq t\leqq s+1.\end{cases}$$

On the other hand,

$$2^{m-u-5+2^{t-1}k}\beta_1^d(2+\beta(u))\beta(u+1)=0$$

by Lemmas 8.5 and 8.10. Thus, we have the desired result. q.e.d.

LEMMA 9.4. *Under the same assumption as in Lemma* 9.3, *we have*

$$\sum_{t=0}^{s+1}2^{m-s-4+2^tk}\beta_1^d\beta(s+1-t)=0.$$

PROOF. By summarizing the relations in Lemma 9.3 over t $(2\leqq t\leqq s+1)$, we have

$$\sum_{t=1}^{s+1}2^{m-s-4+2^tk}\beta_1^d\beta(s+1-t)$$
$$=2^{m-s-2}\beta_1^d\beta(s)^k-2^{m-s-2}\beta_1^{d+2^sk}-2^{m-s-4+2k}\beta_1^d\beta(s)$$
$$=2^{m-s-2}\beta_1^d\beta(s)^k-2^{m-s-4+2k}\beta_1^d\beta(s)\ \text{(by Lemma 8.1).}$$

On the other hand,

$$2^{m-s-2}\beta_1^d\beta(s)^k$$
$$=\sum_{i=0}^{k'}\binom{k'}{i}(-1)^i2^{m-s-2+2i}\beta_1^d\beta(s)^{i+1}\beta(s+1)^{k'-i}\ \text{(by (3.13))}$$
$$=2^{m-s-2}\beta_1^d\beta(s)\beta(s+1)^{k'}+(-1)^{k'}2^{m-s-3+k}\beta_1^d\beta(s)^{k'+1}\ \text{(by Lemma 8.1)}$$
$$=\pm 2^{m-s-5+k}\beta_1^d\beta(s)\beta(s+1)+2^{m-s-4+2k}\beta_1^d\beta(s)\ \text{(by Lemmas 8.1–2)}$$
$$=\pm 2^{m-s-4+k}\beta_1^d\beta(s+1)+2^{m-s-4+2k}\beta_1^d\beta(s)\ \text{(by Lemmas 8.5, 8.10).}$$

Therefore, we have the desired result. q.e.d.

LEMMA 9.5. *Suppose* $2\leqq s\leqq m-2$ *and* $d>0$ *is even under the assumption* (6.1). *Then*

$$2^{m-s-3+k-l}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^l=-2^{m-s-4+k-l}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{l+1}$$

for $0\leqq l\leqq k-2$.

PROOF. By Lemma 3.14, $2^{k-l-2}\beta_1^{d-1}\beta(s)^l P_{m,1}=0$, and so

$$2^{m-s-3+k-l}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^l$$
$$+\sum_{I_s}2^{m-s-3+k-l-j}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^l\beta(i_1)\cdots\beta(i_j)=0,$$

where $I_s=\{(i_1,\cdots,i_j)\colon 1\leqq j\leqq m-s-1,\ s\leqq i_1<\cdots<i_j\leqq m-2\}$.
The terms for $(i_1,\cdots,i_j)\in I_s$ vanish except for $(s)\in I_s$ by Lemma 8.1 in the similar way to the proof of Lemma 6.15. Thus, the desired result follows. q.e.d.

LEMMA 9.6. *Under the same assumption as in Lemma* 9.5, *we have*
$$2^{m-s-3+k}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))=(-1)^{k-1}2^{m-s-2}\beta_1^{d+1}\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1}.$$

PROOF. By Lemma 9.5, we have
$$2^{m-s-3+k}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))=(-1)^{k-1}2^{m-s-2}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1}.$$
On the other hand,
$$2^{m-s-2}\beta_1^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1}$$
$$=2^{m-s-2}\beta_1^{d+1}\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1}+2^{m-s}\beta_1^{d}\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1}\quad\text{(by (3.13))}$$
$$=2^{m-s-2}\beta_1^{d+1}\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1}\quad\text{(by Lemma 8.1)}.$$
Thus, we have the desired result. q.e.d.

LEMMA 9.7. *Under the same assumption as in Lemma* 9.5, *we have*
$$2^{m-s-1}\beta_1^d\beta(s-1)\beta(s)^{k-1}=(-1)^k2^{m-s-4+2k}\beta_1^d\beta(s)-2^{m-s-2}\beta_1^d\beta(s-1)\beta(s)^k.$$

PROOF. By Lemma 3.14, $\beta_1^d\beta(s)^{k-2}P_{m,s}=0$, and so
$$2^{m-s-1}\beta_1^d\beta(s-1)\beta(s)^{k-1}$$
$$=-2^{m-s}\beta_1^d\beta(s)^{k-1}-\sum_{I_s}2^{m-s-1-j}(2+\beta(s-1))\beta_1^d\beta(s)^{k-1}\beta(i_1)\cdots\beta(i_j).$$
The terms for $(i_1,\cdots,i_j)\in I_s$ vanish except for $(s)\in I_s$ by Lemma 8.1, and
$$2^{m-s}\beta_1^d\beta(s)^{k-1}=(-1)^k2^{m-s-4+2k}\beta_1^d\beta(s)\quad\text{(by Lemma 8.2)}.$$
The term for $(s)\in I_s$ is equal to
$$-(2+\beta(s-1))2^{m-s-2}\beta_1^d\beta(s)^k=\pm2^{m-s-3+2k}\beta_1^d\beta(s)-2^{m-s-2}\beta_1^d\beta(s-1)\beta(s)^k$$
(by Lemmas 8.1–2).

Thus, we complete the proof. q.e.d.

LEMMA 9.8. *Under the same assumption as in Lemma* 9.5, *we have*
$$2^{m-s-2}\beta_1^d\beta(s)^k$$
$$=2^{m-s-2}\beta_1^{d+1}\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1}+(-1)^k2^{m-s-4+2k}\beta_1^d\beta(s)$$
$$-2^{m-s-2}\beta_1^d\beta(s-1)\beta(s)^k+\sum_{u=0}^{s-2}2^{m-s-1}\beta_1^d\beta(u)\prod_{t=u+1}^{s-1}(2+\beta(t))\beta(s)^{k-1}.$$

PROOF. In the same manner as the proof of Lemma 6.20, we can prove the lemma by making use of Lemmas 3.14(i) and 9.7. q.e.d.

LEMMA 9.9. *Suppose* $2 \leqq s \leqq m-2$ *and* d *is even under the assumption* (6.1). *Then*

$$2^{m-s-2}\beta_1^d\beta(s)^k = \textstyle\sum_{t=0}^{s} 2^{m-s-4+2^{t+1}k}\beta_1^d\beta(s-t) + (-1)^{k-1}2^{m-s-4+2k}\beta_1^d\beta(s).$$

PROOF. In the case k is even.

$$\begin{aligned} &2^{m-s-2}\beta_1^d\beta(s)^k \\ &= 2^{m-s-2}\beta_1^d\beta(s+1)^{k/2} - 2^{m-s-4+2k}\beta_1^d\beta(s) \quad \text{(by Lemma 9.1)} \\ &= \pm 2^{m-s-4+k}\beta_1^d\beta(s+1) - 2^{m-s-4+2k}\beta_1^d\beta(s) \quad \text{(by Lemmas 8.1–2)}. \end{aligned}$$

In the case k is odd, by the last part of the proof of Lemma 9.4,

$$2^{m-s-2}\beta_1^d\beta(s)^k = \pm 2^{m-s-4+k}\beta_1^d\beta(s+1) + 2^{m-s-4+2k}\beta_1^d\beta(s).$$

Therefore, we have the desired result by Lemmas 9.2 and 9.4. q.e.d.

LEMMA 9.10. *Under the same assumption as in Lemma* 9.9, *we have*

$$2^{m-s-2}\beta_1^d\beta(s-1)\beta(s)^k = \pm 2^{m-s-2+2^2k}\beta_1^d\beta(s-1).$$

PROOF. By Lemma 9.9, we have

$$\begin{aligned} 2^{m-s-2}\beta_1^d\beta(s-1)\beta(s)^k &= (1+(-1)^{k-1})2^{m-s-4+2k}\beta_1^d\beta(s-1)\beta(s) \\ &+ \textstyle\sum_{t=2}^{s} 2^{m-s-4+2^{t+1}k}\beta_1^d\beta(s-t)\beta(s-1) + 2^{m-s-4+2^2k}\beta_1^d\beta(s-1)^2 \\ &= 2^{m-s-4+2^2k}\beta_1^d\beta(s-1)^2 \quad \text{(by Lemma 8.1)} \\ &= 2^{m-s-4+2^2k}\beta_1^d\beta(s) - 2^{m-s-2+2^2k}\beta_1^d\beta(s-1) \quad \text{(by (3.13))} \\ &= \pm 2^{m-s-2+2^2k}\beta_1^d\beta(s-1) \quad \text{(by Lemma 8.1)}. \end{aligned}$$

These complete the proof. q.e.d.

LEMMA 9.11. *Under the same assumption as in Lemma* 9.5, *we have*

$$2^{m-s-1}\beta_1^{d+1}\textstyle\prod_{t=1}^{s-1}(2+\beta(t))\beta(s)^{k-1} = (-1)^{k-1}2^{m-s-2+k}\beta_1^{d-2}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t)).$$

PROOF. By (3.13), we have

$$\begin{aligned} 2^{m-s-1}\beta_1^{d+1}\textstyle\prod_{t=1}^{s-1}(2+\beta(t))\beta(s)^{k-1} &= 2^{m-s-1}\beta_1^{d-2}\beta(1)\textstyle\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1} \\ &- 2^{m-s+1}\beta_1^{d-1}\textstyle\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1} - 2^{m-s}\beta_1^{d}\textstyle\prod_{t=1}^{s-1}(2+\beta(t))\beta(s)^{k-1} \\ &= 2^{m-s-1}\beta_1^{d-2}\beta(1)\textstyle\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{k-1} \quad \text{(by Lemma 8.10)}. \end{aligned}$$

On the other hand,

$$2^{k-l-1}\beta_1^{d-2}\beta(s)^l P_{m,1} = 0 \quad (0 \leqq l \leqq k-2)$$

by Lemma 3.14, and so we have

$$2^{m-s+k-l-2}\beta_1^{d-2}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^l$$
$$+\sum_{I_s}2^{m-s+k-l-2-j}\beta_1^{d-2}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^l\beta(i_1)\cdots\beta(i_j)=0.$$

The terms for $(i_1, \cdots, i_j) \in I_s$ vanish except for $(s) \in I_s$ by Lemma 8.1. Thus

$$2^{m-s+k-l-2}\beta_1^{d-2}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^l=-2^{m-s+k-l-3}\beta_1^{d-2}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\beta(s)^{l+1}$$

for any l with $0 \leqq l \leqq k-2$. Therefore, we have the desired result. q.e.d.

LEMMA 9.12. *Suppose* $3 \leqq s \leqq m-2$ *and* $d>0$ *is even under the assumption* (6.1). *Then*

$$2^{m-s-1}\beta_1^d\beta(u)\prod_{t=u+1}^{s-1}(2+\beta(t))\beta(s)^{k-1}$$
$$=\pm\sum_{l=1}^{s}(-1)^{2^{l-1}}2^{m-s-3+2^{l+1}k}\beta_1^d\beta(u)\prod_{t=u+1}^{s-2}(2+\beta(t))\beta(s-l)$$

for any u *with* $1 \leqq u \leqq s-2$.

PROOF. By Lemma 3.14,

$$\beta_1^d\beta(u)\prod_{t=u+1}^{s-2}(2+\beta(t))\beta(s)^{k-2}P_{m,s}=0,$$

and so

$$2^{m-s-1}\beta_1^d\beta(u)\prod_{t=u+1}^{s-1}(2+\beta(t))\beta(s)^{k-1}$$
$$=-\sum_{I_s}2^{m-s-1-j}\beta_1^d\beta(u)\prod_{t=u+1}^{s-1}(2+\beta(t))\beta(s)^{k-1}\beta(i_1)\cdots\beta(i_j).$$

The terms for $(i_1, \cdots, i_j) \in I_s$ vanish except for $(s) \in I_s$ by Lemma 8.1. Thus, we have

$$2^{m-s-1}\beta_1^d\beta(u)\prod_{t=u+1}^{s-1}(2+\beta(t))\beta(s)^{k-1}$$
$$=\pm 2^{m-s-2}\beta_1^d\beta(u)\prod_{t=u+1}^{s-1}(2+\beta(t))\beta(s)^k \text{ (by Lemma 8.1)}$$
$$=\pm\sum_{l=0}^{s}2^{m-s-3+2^{l+1}k}\beta_1^d\beta(u)\prod_{t=u+1}^{s-2}(2+\beta(t))\beta(s-l)\pm 2^{m-s-3+2k}\beta_1^d\beta(u)\prod_{t=u+1}^{s-2}(2+\beta(t))\beta(s)$$
$$\pm 2^{m-s-2+2^2k}\beta_1^d\beta(u)\prod_{t=u+1}^{s-2}(2+\beta(t))\beta(s-1) \text{ (by Lemmas 8.1, 9.9–10)}$$
$$=\pm\sum_{l=1}^{s}(-1)^{2^{l-1}}2^{m-s-3+2^{l+1}k}\beta_1^d\beta(u)\prod_{t=u+1}^{s-2}(2+\beta(t))\beta(s-l).$$

Therefore, we have the desired result. q.e.d.

LEMMA 9.13. *Under the same assumption as in Lemma* 9.12, *we have*

$$\sum_{l=1}^{s}\sum_{u=1}^{s-2}(-1)^{2^{l-1}}2^{m-s-3+2^{l+1}k}\beta_1^d\beta(u)\beta(s-l)\prod_{t=u+1}^{s-2}(2+\beta(t))=\pm 2^{m-s-2+2^2k}\beta_1^d\beta(s-1).$$

PROOF. We can prove that the left hand side of the desired relation is equal to

$$\pm 2^{m-s-2+2^2k}\beta_1^d\beta(s-1)\pm 2^{m-4+2^sk}\beta_1^d\beta(1)$$

by making use of Lemmas 8.1 and 8.12 instead of Lemmas 5.1 and 5.9 respectively in the proof of Lemma 6.28. While

$$2^{m-4+2^sk}\beta_1^d\beta(1)=0 \text{ (by Lemma 8.1).}$$

Therefore, we have the desired relation. q.e.d.

The following lemma is the immediate consequence of Lemmas 9.11–13.

LEMMA 9.14. *Under the same assumption as in Lemma* 9.5, *we have*

$$2^{m-s-1}\beta_1^d\sum_{u=0}^{s-2}\beta(u)\prod_{t=u+1}^{s-1}(2+\beta(t))\beta(s)^{k-1}$$
$$=(-1)^{k-1}2^{m-s-2+k}\beta_1^{d-2}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))\pm 2^{m-s-2+2^t k}\beta_1^d\beta(s-1).$$

§10. The group $\widetilde{KO}(S^{4n+3}/Q_r)$ $(r=2^{m-1})$ for even n

In this section, we shall determine the additive structure of $\widetilde{KO}(S^{4n+3}/Q_r)$ $(r=2^{m-1})$ with $m\geqq 2$ for even n by giving an additive base. In case $m=1$, $\widetilde{KO}(S^{4n+3}/Q_1)=\widetilde{KO}(L^{2n+1}(4))$ and its additive structure is given in [12, Th. B]. The result in case $m=2$ is given in [7, Th. 1.3].

Let $m\geqq 2$. Then, we have the relations in $\widetilde{KO}(S^{4n+3}/Q_r)$ given by the following propositions.

PROPOSITION 10.1. *Suppose* $2\leqq s\leqq m-2$ *and* $d>0$ *is even under the assumption* (6.1). *Then*

$$2^{m-s-3+k}\beta_1^{d-2}\beta(2)\prod_{t=1}^{s-1}(2+\beta(t))+(-1)^k\sum_{t=0}^{s}(-1)^{2^t}2^{m-s-4+2^{t+1}k}\beta_1^d\beta(s-t)=0.$$

PROOF. The desired relation is the immediate consequence of Lemmas 9.6, 9.8–10 and 9.14. q.e.d.

PROPOSITION 10.2. $2^{n+2}\alpha_0=0$ *and* $2^{n+2}\alpha_1=0$.

PROOF. We see easily that $2^{n+2}\alpha_0=2\alpha_0\beta_1^{n+1}$ in $\widetilde{RO}(Q_r)$ by Propositions 2.5 and 2.7, and $2\alpha_0\beta_1^{n+1}\in\operatorname{Ker}\xi$ by Lemma 3.10.
Therefore, $2^{n+2}\alpha_0=0$ in $\widetilde{KO}(S^{4n+3}/Q_r)$ by (3.9) and the definitions of α_0, $2\beta_1$ and $\beta_1^2\in\widetilde{KO}(S^{4n+3}/Q_r)$ in (3.3) (see also Propositons 2.5 and 2.7). In the similar way to the proof of Proposition 7.4(ii), we have

$$0=2\alpha_1\beta_1^{n+1}=(-1)^n2^{n+2}\sum_{u=1}^{m-2}\beta(u)\prod_{t=u+1}^{m-2}(2+\beta(t))+(-1)^{n+1}2^{n+2}\alpha_1$$

in $\widetilde{KO}(S^{4n+3}/Q_r)$. From this relation, $2^{n+2}\alpha_1=0$ if $m=2$. Let $m\geqq 3$. Then, by Lemma 8.1,

$$2^{n+i}\beta(m-i)=0\ (2\leqq i\leqq m-1).$$

Therefore, we have

$$2^{n+2}\sum_{u=1}^{m-2}\beta(u)\prod_{t=u+1}^{m-2}(2+\beta(t))=0,$$

and so $2^{n+2}\alpha_1=0$. q.e.d.

Now, we are ready to prove Theorem 1.6 for even n.

PROOF OF THEOREM 1.6 FOR EVEN n. The group $\widetilde{KO}(S^{4n+3}/Q_r)$ $(r=2^{m-1})$ for even n is additively generated by α_0, α_1 and $\bar{\delta}_i$ $(1\leqq i\leqq N')$ by Propositions 2.5, 2.7, (3.9), (3.13) and the fact that $2P_{m,1}=\beta_1P_{m,1}=0$ in Lemma 3.14(ii). On the other hand, $2^{n+2}\times 2^{n+2}\times \prod_{i=1}^{N'}\bar{u}(i)=2^{(m+3)n+4}=\#\widetilde{KO}(S^{4n+3}/Q_r)$ by Proposition 4.13 (ii), 7.3, Lemmas 9.2, 9.4, Propositions 10.1–2, Lemma 8.1 and the definitions of $\bar{\alpha}_1$, $\bar{u}(i)$ and $\bar{\delta}_i$ $(1\leqq i\leqq N')$ in (1.5). Therefore, we complete the proof of Theorem 1.6 for even n. q.e.d.

COROLLARY 10.3 (cf. [13, Cor.1.7]). *The order of* $\bar{\delta}_1$ *in* $\widetilde{KO}(S^{4n+3}/Q_r)$ *is equal to* 2^{m+2n-2} *if* n *is an even integer.*

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