宮崎大学教育文化学部紀要　自然科学　第29号（2013）1－9頁

On the Norm Kernel for Three-Cluster System Including a SU(3) Non-Scalar Cluster

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Abstract

A calculational method is extended to three-cluster system including a SU(3) non-scalar cluster in order to study more various cluster systems and is applied to the ^{39}K+α+α and ^{15}N+α+α cluster systems. Eigenvalues and eigenstates of the norm kernel are calculated and discussed.

1. Introduction

Much attention has been devoted to superdeformed (SD) rotational bands in ^{36}Ar-^{48}Cr nuclei [1, 2]. These collective aspects are particularly interesting subjects for nuclear structure studies. I have performed α-cluster model calculations for ^{36}Ar-^{48}Cr and have shown that the α-cluster structure is a stable feature in this region [3]. The relationship between the SD bands and the 2α-cluster structure has also been discussed in several studies [4-7], largely because the SD structure of these nuclei is regarded as a key to understanding the relationship between the SD bands and the cluster structure. Therefore, it is very interesting to apply the multi-cluster model approach, which can treat various cluster configurations, to the SD bands in this region. This application could offer valuable insight into the SD bands and their strong collectivity. In this paper, I will present an extension of the previous calculational method [8] for determining the norm kernel of three cluster systems in order to study a wider variety of cluster systems.

2. The norm kernel

In this article, I mainly discuss the ^{39}K+α+α system as an example of the explanation of the present method. The system provides the first link in the chain of 2α-*nh* cluster states in A=44-48 nuclei and is well suited for this purpose. Moreover, it is not so difficult to apply the method to ^{15}N+α+α and other cluster systems. In the ^{39}K+α+α cluster system, the model space is described by a set of wave functions

$$\Psi\big((N_{23}\,l_{23},\,N_1\,l_1)L_R,\,(l_h,\,1/2)I_h;\,JM\big)$$
$$=\sqrt{\frac{4!4!39!}{47!}}\,\mathcal{A}\Big\{\phi^{\rm int}(\alpha)\phi^{\rm int}(\alpha)\Big[\phi^{\rm int}_{Ih(lh,1/2)}(^{39}{\rm K})\big[U_{N_{23},l_{23}}(\mathbf{r}),\,U_{N_1,l_1}(\mathbf{R})\big]_{L_R}\Big]_{JM}\Big\}. \tag{1}$$

The $\phi^{\rm int}(\alpha)$ is the antisymmetrized internal wave function of α and corresponds to $(0s)^4$ configuration.

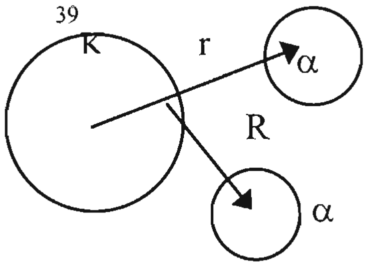

Fig. 1. Coordinates of the ^{39}K+α+α system.

The core state $\phi^{\rm int}_{Ih(lh,1/2)}(^{39}\mathrm{K})$ is assumed to be (λ,μ) = (0,2) state according to removing one nucleon from the sd-orbits in ^{40}Ca. The orbital angular momentum l_h and the spin 1/2 of the ^{39}K are coupled to the total angular momentum I_h. We adopt a common oscillator constant a for all clusters. The relative wave functions $U_{N23,l23}(r)$ and $U_{N1,l1}(R)$ are harmonic oscillator (H.O.) wave functions with $N_{23}l_{23}$ and $N_1 l_1$ quantum numbers. The relative coordinates r and R are shown in Fig.1. The bracket implies two angular momenta's coupling to a resultant angular momentum. Therefore, the relative angular momenta, l_{23} and l_1, are coupled to the total relative angular momentum L_R. Furthermore, I_h and L_R are coupled to the total angular momentum J of ^{47}V nucleus.

The norm kernel is concerned essentially with the symmetry of the spatial part of the cluster wave functions, i.e., the total orbital angular momentum L is a good quantum number. Therefore, it is convenient to use the following L-projected wave functions to calculate the norm kernel. We use the model wave function

$$\Phi\big((N_{23}\,l_{23},N_1\,l_1)L_R,l_h;LM_L\big)$$
$$=\sqrt{\frac{4!4!39!}{47!}}\mathcal{A}\Big\{\phi^{\rm int}(\alpha)\phi^{\rm int}(\alpha)\Big[\big[U_{N_{23},l_{23}}(\mathbf{r}),\,U_{N_1,l_1}(\mathbf{R})\big]_{L_R},\phi^{\rm int}_{l_h}(^{39}\mathrm{K})\Big]_{LM_L}\Big\}. \tag{2}$$

In the above equation, the relative angular momentum and the orbital angular momentum of ^{39}K, L_R and l_h, respectively, are coupled to the total orbital angular momentum L. The relationship between the J-projected set of states of Eq.(1) and the L-projected set in Eq.(2) can be derived easily in angular momentum recoupling. Hereafter, we discuss the method by using the mainly L-projected wave functions. The model wave function is generated as a direct product of the two relative wave functions and a core wave function : $(N_{23},0)\times(N_1,0)\times(\lambda_h,\mu_h)$. The Pauli-allowed states are obtained by diagonalizing the norm kernel.

$$\Sigma_{N_{23}'l_{23}',N_1'l_1'L_R}\Big\langle\phi^{\rm int}(\alpha)\phi^{\rm int}(\alpha)\Big[\big[U_{N_{23},l_{23}}(\mathbf{r}),U_{N_1,l_1}(\mathbf{R})\big]_{L_R},\phi^{\rm int}_{l_h}(^{39}\mathrm{K})\Big]_{LM_L}$$
$$\mathcal{A}\Big\{\phi^{\rm int}(\alpha)\phi^{\rm int}(\alpha)\Big[\big[U_{N_{23}',l_{23}'}(\mathbf{r}),U_{N_1',l_1'}(\mathbf{R})\big]_{L_R},\phi^{\rm int}_{l_h}(^{39}\mathrm{K})\Big]_{LM_L}\Big\}\Big\rangle C^{LNQ}_{(N_{23}'l_{23}',N_1',l_1')L_R,l_h} \tag{3}$$
$$=\mu^{NQ}C^{LNQ}_{(N_{23}l_{23},N_1l_1)L_R,l_h}.$$

Thus the totally antisymmetrized basis that satisfies the orthonormal condition is given by

$$\Phi^{LNQ} = \frac{1}{\sqrt{\mu^{NQ}}}\Sigma_{(N_{23}l_{23},N_1 l_1)L_R,l_h} C^{LNQ}_{(N_{23}l_{23},N_1 l_1)L_R,l_h}\Phi\big((N_{23}l_{23},N_1 l_1)L_R,l_h;LM_L\big). \tag{4}$$

The eigenstates are classified by the total oscillator quanta $N=N_{23}+N_1$ and the SU(3) symmetry. The symbol Q denotes the SU(3) quantum number $(\lambda,\mu)K$.

Then the problem is to calculate the norm kernel and the eigenvalues μ^{NQ}. It is very tedious and enormous to treat the antisymmetrization operator within the internal and relative coordinate system. Therefore the computation of the kernels in heavy systems is done by calculating the corresponding GCM kernels and transforming them. This can be done by noting the fact that corresponding GCM kernel plays the role of a generating function of the kernel of Eq.(3).

3. The norm kernel in GCM space

For the sake of an explanation, I use a more general notation in this section. We treat the GCM wave function of the form

$$\Phi^{GCM}(\mathrm{S}_1,\mathrm{S}_2,\mathrm{S}_3)=\sqrt{\frac{\mathrm{A}_1!\mathrm{A}_2!\mathrm{A}_3!}{\mathrm{A}!}}\mathcal{A}\{\phi(\mathrm{A}_1,\mathrm{S}_1)\phi(\mathrm{A}_2,\mathrm{S}_2)\phi(\mathrm{A}_3,\mathrm{S}_3)\}, \tag{5}$$

where $\mathcal{A}$ is the antisymmetrizer which exchanges the nucleons belonging to different clusters and A_i is the mass number of the cluster i and specifies the cluster. The total mass number is $\mathrm{A}=\mathrm{A}_1+\mathrm{A}_2+\mathrm{A}_3$. For the present system we simply take $\mathrm{A}_1=\mathrm{A}_2=4$ and $\mathrm{A}_3=39$. The antisymmetrized cluster wave function $\phi(\mathrm{A}_i,\mathrm{S}_i)$ is assumed to be $(0s)^4$ and $(0s)^4(0p)^{12}(sd)^{23}$ shell-model wave functions centered at $\boldsymbol{S}i$ for α and ^{39}K, respectively. The center-of-mass (C.M.) point of the total system is typically chosen as the origin O, that is $\mathrm{A}_1\mathrm{S}_1+\mathrm{A}_2\mathrm{S}_2+\mathrm{A}_3\mathrm{S}_3=0$. Since $\phi(\mathrm{A}_i,\mathrm{S}_i)$ is a Slater determinant, $\Phi^{GCM}(\mathrm{S}_1,\mathrm{S}_2,\mathrm{S}_3)$ is also a Slater determinant. This fact makes the calculation by GCM straightforward.

The shell-model wave function of each cluster can be separated into the internal and the C.M. wave functions.

$$\phi(\mathrm{A}_i,\mathrm{S}_i) = \phi^{\mathrm{int}}(\mathrm{A}_i)\Big[\frac{a\mathrm{A}_i}{\pi}\Big]^{\frac{3}{4}}\exp\Big\{-\frac{a\mathrm{A}_i}{2}(\mathrm{X}_i-\mathrm{S}_i)^2\Big\}, \tag{6}$$

where $\boldsymbol{X}_i$ is the C.M. coordinate of the cluster A_i. Substituting this equation into Eq.(5) gives

$$\Phi^{GCM}(\mathrm{S}_1,\mathrm{S}_2,\mathrm{S}_3)=\sqrt{\frac{\mathrm{A}_1!\mathrm{A}_2!\mathrm{A}_3!}{\mathrm{A}!}}\prod_{i=1}^{3}\Big[\frac{a\mathrm{A}_i}{\pi}\Big]^{\frac{3}{4}}\mathcal{A}\Big[\exp\Big\{-\sum_{i=1}^{3}\frac{a\mathrm{A}_i}{2}(\mathrm{X}_i-\mathrm{S}_i)^2\Big\}\phi^{\mathrm{int}}(\mathrm{A}_1)\phi^{\mathrm{int}}(\mathrm{A}_2)\phi^{\mathrm{int}}_{lhmh}(\mathrm{A}_3)\Big]. \tag{7}$$

When we treat the system with definite angular momentum, we need to project out the definite relative angular momentum components from $\Phi^{GCM}(\mathrm{S}_1,\mathrm{S}_2,\mathrm{S}_3)$. In order to observe the relative motions of the clusters, it is convenient to use the inter-cluster relative coordinates ($\boldsymbol{r}$ and $\boldsymbol{R}$) and the total C.M. coordinate $\boldsymbol{X}_G$.

$$\Phi^{GCM}_{l_h m_h}(\mathbf{d},\mathbf{D}) = \Big(\frac{a\mathrm{A}}{\pi}\Big)^{\frac{3}{4}}\exp\Big\{-\frac{a\mathrm{A}}{2}\mathrm{X}_G^2\Big\}\sqrt{\frac{\mathrm{A}_1!\mathrm{A}_2!\mathrm{A}_3!}{\mathrm{A}!}}$$

$$\times\mathcal{A}\Big[\Big(\frac{a_1}{\pi}\Big)^{\frac{3}{4}}\exp\Big\{-\frac{a_1}{2}(\mathrm{R}-\mathrm{D})^2\Big\}\Big(\frac{a_{23}}{\pi}\Big)^{\frac{3}{4}}\exp\Big\{-\frac{a_{23}}{2}(\mathbf{r}-\mathbf{d})^2\Big\}\phi^{\mathrm{int}}(\mathrm{A}_1)\phi^{\mathrm{int}}(\mathrm{A}_2)\phi^{\mathrm{int}}_{l_h m_h}(\mathrm{A}_3)\Big], \tag{8}$$

where $a_1 = \frac{A_1(A_2+A_3)}{A} a$, $a_{23} = \frac{A_2A_3}{A_2+A_3} a$ and the displacement parameters of the cluster centers are

$$\mathbf{d}=\mathbf{S}_2-\mathbf{S}_3 \text{ and } \mathbf{D}=\mathbf{S}_1-\frac{A_2}{A_2+A_3}\mathbf{S}_2-\frac{A_3}{A_2+A_3}\mathbf{S}_3. \tag{9}$$

In this way, the dependence on X_G is factored out, and the GCM wave function is a non-spurious wave function about the C.M. motion.

We can evaluate the GCM norm kernel analytically, because $\Phi^{GCM}_{l_h,m_h}(\mathbf{d},\mathrm{D})$ is also a Slater determinant. As cluster A_3 is a non-scalar cluster, the calculation of the kernel is somewhat complicated. The GCM kernel turns out to be given by

$$\begin{aligned}\left\langle\Phi^{GCM}_{lhmh}(\mathbf{d},\mathbf{D})\middle|\Phi^{GCM}_{lh'mh'}(\mathbf{d}',\mathbf{D}')\right\rangle &= \exp\left\{-\frac{a_1}{4}(D^2+D'^2)-\frac{a_{23}}{4}(d^2+d'^2)\right\}\\ &\times\exp\left\{-\frac{aA_1^2}{2A}\mathbf{D}\cdot\mathbf{D}'-\frac{aA}{2}\left[\frac{A_2}{A_2+A_3}\right]^2\mathbf{d}\cdot\mathbf{d}'-\frac{aA_1A_2}{2(A_2+A_3)}(\mathbf{D}\cdot\mathbf{d}'+\mathbf{d}\cdot\mathbf{D}')\right\}\\ &\times\left[\delta_{lh,lh'}\delta_{mh,mh'}|\mathbf{B}|^4+(-)^{mh+mh'}|\mathbf{B}|^3C_{2,lh}C_{2,lh'}\left(\frac{a}{4}\right)^2Z\right],\end{aligned} \tag{10}$$

$$C_{Nl}=(-1)^n\sqrt{\frac{2^{n+l+2}\pi}{n!(2n+2l+1)!}} \text{ and } N=2n+l, \tag{11}$$

The term Z is derived from the overlap integral between the hole states of A_3 cluster and the particle states of A_1 (or A_2) cluster and depends on the angular part of the displacement parameters as follows:

$$\begin{aligned}Z &= y_{lh-mh}(\mathbf{d})y^*_{lh'-mh'}(\mathbf{d}')d^{2-lh}d'^{2-lh'}\varepsilon3\left(\frac{a}{2}\mathbf{t}\cdot\mathbf{t}'\right)\\ &+y_{lh-mh}(\mathbf{t})y^*_{lh'-mh'}(\mathbf{t}')t^{2-lh}t'^{2-lh'}\varepsilon3\left(\frac{a}{2}\mathbf{d}\cdot\mathbf{d}'\right)\\ &-y_{lh-mh}(\mathbf{d})y^*_{lh'-mh'}(\mathbf{t}')d^{2-lh}t'^{2-lh'}\varepsilon3\left(\frac{a}{2}\mathbf{t}\cdot\mathbf{d}'\right)\\ &-y_{lh-mh}(\mathbf{t})y^*_{lh'-mh'}(\mathbf{d}')t^{2-lh}d'^{2-lh'}\varepsilon3\left(\frac{a}{2}\mathbf{d}\cdot\mathbf{t}'\right),\end{aligned} \tag{12}$$

where **t** and **t'** are the displacement parameters between A_1 and A_3 clusters:

$$\mathbf{t}\equiv\mathbf{D}+\frac{A_2}{A_2+A_3}\mathbf{d} \quad\text{and}\quad \mathbf{t}'\equiv\mathbf{D}'+\frac{A_2}{A_2+A_3}\mathbf{d}' \tag{13}$$

In Eq.(12) the abbreviated notation $\varepsilon3(x)\equiv\exp(x)-1-x-x^2/2$ are used. The determinant $|\mathbf{B}|$ is coming from the direct part of the hole states of the A_3 cluster and is written as follows:

$$|\mathbf{B}| = \varepsilon3(\rho\mathbf{t}\cdot\mathbf{t}')\varepsilon3(\rho\mathbf{d}\cdot\mathbf{d}')-\varepsilon3(\rho\mathbf{d}\cdot\mathbf{t}')\varepsilon3(\rho\mathbf{t}\cdot\mathbf{d}') \tag{14}$$

with $\rho=a/2$. The determinant $|\mathbf{B}|$ can be written as a sum of terms, each of which is a product of polynomial parts and Gaussian parts.

$$\begin{aligned}|\mathbf{B}| &= \sum_{i=1}^{max} g(i)(\rho\mathbf{D}\cdot\mathbf{D}')^{k1(i)}(\rho\mathbf{D}\cdot\mathbf{d}')^{k2(i)}(\rho\mathbf{d}\cdot\mathbf{D}')^{k3(i)}(\rho\mathbf{d}\cdot\mathbf{d}')^{k4(i)}\\ &\times\exp\left\{\omega1(i)\rho\mathbf{D}\cdot\mathbf{D}'+\omega2(i)\rho\mathbf{D}\cdot\mathbf{d}'+\omega3(i)\rho\mathbf{d}\cdot\mathbf{D}'+\omega4(i)\rho\mathbf{d}\cdot\mathbf{d}'\right\}.\end{aligned} \tag{15}$$

We can see that four types of scalar products of the displacement parameters are included in the determinant.

In the GCM norm kernel of Eq.(10), the two types of the determinants, $|\mathbf{B}|^3$ and $|\mathbf{B}|^4$, are included. The $|\mathbf{B}|^4$ term is generated from the direct part of the overlap kernel between the 2α and A_3 cluster wave functions. We call the term an 8p-1h type. The $|\mathbf{B}|^3$ term is generated from the one-nucleon exchange part of overlap kernel between 2α and A_3 cluster wave functions and is called a 7p type. Using Eq.(15), the $|\mathbf{B}|^4$ term can be expressed as

$$\begin{aligned}|\mathrm{B}|^4 =& \sum_{i1=1}^{\max} g(i1) \sum_{i2=1}^{\max} g(i2) \sum_{i3=1}^{\max} g(i3) \sum_{i4=1}^{\max} g(i4) (\rho\mathbf{D}\cdot\mathbf{D}')^{k1(i1)+k1(i2)+k1(i3)+k1(i4)} \\ &\times (\rho\mathbf{D}\cdot\mathbf{d}')^{k2(i1)+k2(i2)+k2(i3)+k2(i4)} (\rho\mathbf{d}\cdot\mathbf{D}')^{k3(i1)+k3(i2)+k3(i3)+k3(i4)} \\ &\times (\rho\mathbf{D}\cdot\mathbf{d}')^{k4(i1)+k4(i2)+k4(i3)+k4(i4)} \\ &\times \exp\left\{\left(\omega1(i1)+\omega1(i2)+\omega1(i3)+\omega1(i4)\right)\rho\mathbf{D}\cdot\mathbf{D}'\right\} \\ &\times \exp\left\{\left(\omega2(i1)+\omega2(i2)+\omega2(i3)+\omega2(i4)\right)\rho\mathbf{D}\cdot\mathbf{d}'\right\} \\ &\times \exp\left\{\left(\omega3(i1)+\omega3(i2)+\omega3(i3)+\omega3(i4)\right)\rho\mathbf{d}\cdot\mathbf{D}'\right\} \\ &\times \exp\left\{\left(\omega4(i1)+\omega4(i2)+\omega4(i3)+\omega4(i4)\right)\rho\mathbf{d}\cdot\mathbf{d}'\right\}.\end{aligned} \tag{16}$$

By recoupling the vectors, the polynomial parts of the scalar products can be transformed to tensor coupled terms of the four displacement vectors.

$$\begin{aligned}(\mathbf{D}\cdot\mathbf{D}')^{k1}(\mathbf{D}\cdot\mathbf{d}')^{k2}(\mathbf{d}\cdot\mathbf{D}')^{k3}(\mathbf{d}\cdot\mathbf{d}')^{k4} =& (4\pi)^2 \sum_{p1=0}^{[k1/2]} \sum_{p2=0}^{[k2/2]} \sum_{p3=0}^{[k3/2]} \sum_{p4=0}^{[k4/2]} (-)^{k1+k2+k3+k4} \\ &\times \left\{A^{k1}_{k1-2p1} A^{k2}_{k2-2p2} A^{k3}_{k3-2p3} A^{k4}_{k4-2p4}\right\}^2 \\ &\times \sum_{a3}\sum_{a4}\sum_{a5}\sum_{a6} (k2-2p2,0,k1-2p1,0\,|\,a4,0)(k4-2p4,0,k3-2p3,0\,|\,a3,0) \\ &\times (k4-2p4,0,k2-2p2,0\,|\,a5,0)(k3-2p3,0,k1-2p1,0\,|\,a6,0) \\ &\times \sum_{l}\sqrt{2l+1} \times \begin{Bmatrix} k4-2p4 & k2-2p2 & a5 \\ k3-2p3 & k1\ 2p1 & a6 \\ a3 & a4 & l \end{Bmatrix} \left[\left[Y_{a3}(\hat{\mathbf{d}}), Y_{a4}(\hat{\mathbf{D}})\right]_l, \left[Y_{a5}(\hat{\mathbf{d}}'), Y_{a6}(\hat{\mathbf{D}}')\right]_l\right]_0\end{aligned} \tag{17}$$

where the coefficient

$$A_l^N = (-1)^{(N-l)/2}\sqrt{\frac{N!(2l+1)}{(N-1)!!(N+l+1)!!}}. \tag{18}$$

and also the coefficients (|) and { } are the Clebsch-Gordan and the 9j-symboles, respectively.

The GCM wave function with the definite angular-momentum is written in terms of the projections of the relative angular-momenta l_1 and l_{23}.

$$\begin{aligned}\Phi^{GCM}\left((l_{23,}l_1)L_R,l_h{:}LM_L\right) =& \Sigma_{m23,m1,mh}(l_{23}m_{23}l_1m_1\,|\,L_RM_R)(L_RM_Rl_hm_h\,|\,LM_L) \\ &\times \int d\hat{\mathbf{D}}\int d\hat{\mathbf{d}}\, Y_{l1,m1}(\hat{\mathbf{D}})\, Y_{l23,m23}(\hat{\mathbf{d}}) \times \Phi^{GCM}_{llhmh}(\mathbf{d},\mathbf{D}).\end{aligned} \tag{19}$$

The GCM norm kernel with definite angular-momentum turns out to be

$$\begin{aligned}&\left\langle\Phi^{GCM}\big((l_{23},l_1)L_R,l_h:LM_L\big)\middle|\Phi^{GCM}\big((l_{23}',l_1')L_R',l_h':L'M_L'\big)\right\rangle=\delta_{LL'}\,\delta_{M_LM_L'}\\&\times\Sigma_{m_{23},m_1,m_h}\Sigma_{m_{23}',m_1',m_h'}(l_{23}m_{23}l_1m_1|L_RM_R)(L_RM_Rl_hm_h|LM_L)\\&\times(l_{23}'m_{23}'l_1'm_1'|L_R'M_R')(L_R'M_R'l_h'm_h'|L'M_L')\\&\times\int d\hat{\mathbf{D}}\int d\hat{\mathbf{d}}\,Y_{l_1,m_1}^{*}(\hat{\mathbf{D}})\,Y_{l_{23},m_{23}}^{*}(\hat{\mathbf{d}})\times\int d\hat{\mathbf{D}}'\int d\hat{\mathbf{d}}'\,Y_{l_1',m_1'}^{*}(\hat{\mathbf{D}}')\,Y_{l_{23}',m_{23}'}^{*}(\hat{\mathbf{d}}')\\&\times\left\langle\Phi_{l_hm_h}^{GCM}(\mathbf{d},\mathbf{D})\middle|\Phi_{l_h'm_h'}^{GCM}(\mathbf{d}',\mathbf{D}')\right\rangle.\end{aligned}\tag{20}$$

The integrand is a multiplication of the scalar products of the parameters **d, D, d'** and **D'** by the spherical harmonics. We are therefore able to calculate the integrations analytically. In the angular momentum projection, we use the intrinsic overlap integral of Eq.(10), and the kernel is written as a sum of two types of terms, 8p-1h and 7p.

The 8p-1h term contains the $|\mathbf{B}|^2$ determinant and can be expanded by polynomials of the scalar products. Then the 8p-1h term can be calculated analytically.

$$\begin{aligned}G_{8p\text{-}1h}=&\int d\hat{\mathbf{D}}\int d\hat{\mathbf{d}}\,Y_{l_1,m_1}^{*}(\hat{\mathbf{D}})\,Y_{l_{23},m_{23}}^{*}(\hat{\mathbf{d}})\times\int d\hat{\mathbf{D}}'\int d\hat{\mathbf{d}}'\,Y_{l_1',m_1'}(\hat{\mathbf{D}}')\,Y_{l_{23}',m_{23}'}(\hat{\mathbf{d}}')\\&\times(\mathbf{D}\cdot\mathbf{D}')^{k1}(\mathbf{D}\cdot\mathbf{d}')^{k2}(\mathbf{d}\cdot\mathbf{D}')^{k3}(\mathbf{d}\cdot\mathbf{d}')^{k4}.\end{aligned}\tag{21}$$

The 7p term contains the scalar products and also the products of spherical tensors, and is calculated in the following integral:

$$\begin{aligned}G_{7p}=&\int d\hat{\mathbf{D}}\int d\hat{\mathbf{d}}\,Y_{l_1,m_1}^{*}(\hat{\mathbf{D}})\,Y_{l_{23},m_{23}}^{*}(\hat{\mathbf{d}})\times\int d\hat{\mathbf{D}}'\int d\hat{\mathbf{d}}'\,Y_{l_1',m_1'}(\hat{\mathbf{D}}')\,Y_{l_{23}',m_{23}'}(\hat{\mathbf{d}}')\\&\times(\mathbf{D}\cdot\mathbf{D}')^{k1}(\mathbf{D}\cdot\mathbf{d}')^{k2}(\mathbf{d}\cdot\mathbf{D}')^{k3}(\mathbf{d}\cdot\mathbf{d}')^{k4}\times(-)^{mh+mh'}\\&\times\Big[Y_{lh,mh}(\hat{\mathbf{d}})\,Y_{lh',mh'}^{*}(\hat{\mathbf{d}}')\mathrm{d}^2\mathrm{d}'^2\{\varepsilon3(\rho\mathbf{t}\cdot\mathbf{t}')+\gamma^4\varepsilon3(\rho\mathbf{d}\cdot\mathbf{d}')-\gamma^2\varepsilon3(\rho\mathbf{t}\cdot\mathbf{d}')-\gamma^2\varepsilon3(\rho\mathbf{d}\cdot\mathbf{t}')\}\\&\quad+Y_{lh,-mh}(\hat{\mathbf{D}})\,Y_{lh',-mh'}^{*}(\hat{\mathbf{D}}')\mathrm{D}^2\mathrm{D}'^2\varepsilon3(\rho\mathbf{d}\cdot\mathbf{d}')\\&\quad+Y_{lh,mh}(\hat{\mathbf{D}})\,Y_{lh',-mh'}^{*}(\hat{\mathbf{d}}')\mathrm{D}^2\mathrm{d}'^2\{-\varepsilon3(\rho\mathbf{d}\cdot\mathbf{t}')+\gamma^2\varepsilon3(\rho\mathbf{d}\cdot\mathbf{d}')\}\\&\quad+Y_{lh,-mh}(\hat{\mathbf{d}})\,Y_{lh',mh'}^{*}(\hat{\mathbf{D}}')\mathrm{d}^2\mathrm{D}'^2\{-\varepsilon3(\rho\mathbf{t}\cdot\mathbf{d}')+\gamma^2\varepsilon3(\rho\mathbf{d}\cdot\mathbf{d}')\}\Big].\end{aligned}\tag{22}$$

The spherical tensors of the kernel and the Y-functions of angular momentum projection can be recoupled to spherical tensors of each GCM parameter. Therefore the 7p term is also reduced to the integral like Eq.(21) and can be calculated analytically.

On the other hand, the GCM wave function is a generating function of the H.O. wave functions, and it can be expanded by the H.O. wave functions of Eq.(2).

$$\begin{aligned}\Phi^{GCM}\big((l_{23},l_1)L_R,l_h:LM_L\big)=&\sum_{N1}\sum_{N23}C_{N_1,l_1}C_{N_{23},l_{23}}\Big(\frac{\sqrt{a_1}}{2}D\Big)^{N1}\Big(\frac{\sqrt{a_{23}}}{2}d\Big)^{N23}\\&\times\exp\Big(-\frac{a_{23}}{4}d^2-\frac{a_1}{4}D^2\Big)\Phi\big((N_{23}l_{23},N_1l_1)L_R,l_h:LM_L\big)\Phi_{0S}(X_G).\end{aligned}\tag{23}$$

Using this relationship, we are able to derive the relationship between the norm kernel in the GCM wave functions of Eq.(20) and the one in the H.O. basis of Eq.(2).

$$\begin{aligned}
&\left\langle \Phi^{GCM}\big((l_{23}, l_1)L_R, l_h : LM_L\big) \middle| \Phi^{GCM}\big((l_{23}', l_1')L_R', l_h' : L'M_L'\big)\right\rangle = \delta_{LL'}\,\delta_{M_L M_L'} \\
&\times \Sigma_{N1,N23}\,\Sigma_{N1',N23'}\,C_{N_1,l_1}C_{N_{23},l_{23}}C_{N_1',l_1'}C_{N_{23}',l_{23}'}\left(\frac{\sqrt{a_1}}{2}\right)^{N_1+N_1'}\left(\frac{\sqrt{a_{23}}}{2}\right)^{N_{23}+N_{23}'} \\
&\times \exp\left\{-\frac{a_{23}}{4}(d^2+d'^2)-\frac{a_1}{4}(D_1^{\,2}+D_1'^{\,2})\right\}D^{N_1}d^{N_{23}}D'^{N_1'}d'^{N_{23}'} \\
&\times \left\langle \Phi\big((N_{23}l_{23}, N_1 l_1)L_R, l_h : LM_L\big) \middle| \Phi\big((N_{23}'l_{23}', N_1'l_1')L_R', l_h' : L'M_L'\big)\right\rangle
\end{aligned} \tag{24}$$

This relationship gives the transformation procedure from the GCM kernel of Eq.(20) to the kernel in the H.O. basis. The matrix elements of the norm kernel in the H.O. basis are obtained as the coefficients of the powers of d, D, d' and D'. Thus we are able to solve the eigenvalue problem of the norm kernel, which gives all of the necessary quantities.

4. Eigenstates of the norm kernel

The calculated allowed states of ^{39}K+α+α system are listed in Table I, all of which are classified according to the SU(3) symmetry. The states with the total quanta $N \leqq 22$ are not allowed as a matter of course. We can see the lowest N=23 space contains the important $(fp)^7$ shell-model states as (λ, μ) = (15,3), (13,4), (11,5), ⋯ It should be noted that the ^{39}K+α+α model can describe important ground configuration of ^{47}V. The N=24 space includes the $(sd)^{-1}(fp)^8$ states such as (λ, μ) = (18,2), (17,4), (16,6), ⋯ The configurations with a larger value of $N \geqq 25$ have a capacity for presenting the 2α- as well as the α-cluster states. As the number of quanta becomes larger, multiplicity of the same (λ, μ)-states increases notably. It is interesting that even when the number of quanta is large enough and the largest eigenvalue almost tends to unity, there still remain some states with rather small eigenvalues. It is noted that the anti-symmetrizing effect of the three cluster system is still important at large N states

Table I. The Pauli-allowed states of the ^{39}K+α+α system.
They are classified by the SU(3) label (λ,μ) with the multiplicity n.

N	$(\lambda,\mu)^n$
23	$(15,3)(13,4)(11,5)(9,6)(7,7)(5,8)(3,9)$
24	$(18,2)^2(16,3)^2(14,4)^3(12,5)^2(10,6)^3(8,7)^2(6,8)^3(4,9)^2(2,10)^2$ $(17,4)(15,5)(13,6)(11,7)(9,8)(7,9)(5,10)(3,11)(1,12)$ $(16,6)(12,8)(8,10)(4,12)(0,14)$
25	$(21,1)^2(19,2)^3(17,3)^4(15,4)^4(13,5)^4(11,6)^4(9,7)^4(7,8)^4(5,9)^4(3,10)^3(1,11)^2$ $(20,3)(18,4)^2(16,5)^2(14,6)^2(12,7)^2(10,8)^2(8,9)^2(6,10)^2(4,11)^2(2,12)^2(0,13)$ $(19,5)(17,6)(15,7)(13,8)(11,9)(9,10)(7,11)(5,12)(3,13)(1,14)$
26	$(24,0)^2(22,1)^3(20,2)^5(18,3)^5(16,4)^6(14,5)^5(12,6)^6(10,7)^5(8,8)^6(6,9)^5(4,10)^5(2,11)^3(0,12)^2$ $(23,2)(21,3)^2(19,4)^3(17,5)^4(15,6)^3(13,7)^3(11,8)^3(9,9)^3(7,10)^3(5,11)^3(3,12)^3(1,13)^2$ $(22,4)(20,5)(18,6)^2(16,7)(14,8)^2(12,9)(10,10)^2(8,11)(6,12)^2(4,13)(2,14)^2$
27	$(25,0)^3(23,1)^5(21,2)^6(19,3)^7(17,4)^7(15,5)^7(13,6)^7(11,7)^7(9,8)^7(7,9)^7(5,10)^6(3,11)^5(1,12)^3$ $(26,1)(24,2)^2(22,3)^3(20,4)^4(18,5)^4(16,6)^4(14,7)^4(12,8)^4(10,9)^4(8,10)^4(6,11)^4(4,12)^4(2,13)^3(0,14)$ $(25,3)(23,4)(21,5)^2(19,6)^2(17,7)^2(15,8)^2(13,9)^2(11,10)^2(9,11)^2(7,12)^2(5,13)^2(3,14)^2(1,15)$

28	(26,0)^{5}(24,1)^{6}(22,2)^{8}(20,3)^{8}(18,4)^{9}(16,5)^{8}(14,6)^{9}(12,7)^{8}(10,8)^{9}(8,9)^{8}(6,10)^{8}(4,11)^{6}(2,12)^{5}(0,13) (27,1)^{2}(25,2)^{4}(23,3)^{4}(21,4)^{5}(19,5)^{5}(17,6)^{5}(15,7)^{5}(13,8)^{5}(11,9)^{5}(9,10)^{5}(7,11)^{5}(5,12)^{5}(3,13)^{4}(1,14)^{2} (28,2)(26,3)(24,4)^{5}(22,5)(20,6)^{3}(18,7)^{2}(16,8)^{3}(14,9)^{2}(12,10)^{3}(10,11)^{2}(8,12)^{3}(6,13)^{2}(4,14)^{3}(2,15)(0,16)
···	···
N odd	(N-2,0)(N-4,1)(N-6,2)...(3,(N-5)/2)(1,(N-3)/2) (N-1,1)(N-3,2)(N-5,3)...(2,(N-1)/2)(0,(N+1)/2) (N,2)(N-2,3)(N-4,4)...(3,(N+1)/2)(1,(N+3)/2)
N even	(N-2,0)(N-4,1)(N-6,2)...(2,N/2-2)(0,N/2-1) (N-1,1)(N-3,2)(N-5,3)...(3,N/2-1)(1,N/2) (N,2)(N-2,3)(N-4,4)...(2,N/2+1)(0,N/2+2)

For comparison, the allowed states of the ^{15}N+α+α system are listed in Table II. This system provides another testing ground for the persistence of the present method. Furthermore, this is suited for the study of 2α-*nh* cluster states in the A=20-24 sd nuclei. The states with the total quanta $N \leqq 14$ are not allowed in this system. We can see that the ^{15}N+ α + α cluster model space contain many important shell-model states. The lowest quanta N=15 states are the (sd)^{7} ground configurations as (λ, μ) = (8,3), (6,4), (4,5), ··· The important core-excited states such as the (sd)^{8}(p)$^{-1}$ N=16 (λ, μ) = (11,2) and (8,5) are included in the present model space.

Table II. The Pauli-allowed states of the ^{15}N+α+α system.
They are classified by the SU(3) label (λ, μ) with the multiplicity n.

N	$(\lambda, \mu)^{n}$
15	(8,3)(6,4)(4,5)(2,6)
16	(11,2)(9,3)^{2}(7,4)^{2}(5,5)^{2}(3,6)^{2}(1,7) (8,5) (4,7)(0,9)
17	(14,1)(12,2)^{2}(10,3)^{3}(8,4)^{3}(6,5)^{3}(4,6)^{3}(2,7)^{2}(0,8) (11,4)(9,5)(7,6)(5,7)(3,8)(1,9)
18	(17,0)(15,1)^{2}(13,2)^{3}(11,3)^{4}(9,4)^{4}(7,5)^{4}(5,6)^{4}(3,7)^{3}(1,8)^{2} (14,3)(12,4)(10,5)^{2}(8,6)(6,7)^{2}(4,8)(2,9)^{2}
19	(18,0)^{2}(16,1)^{3}(14,2)^{4}(12,3)^{5}(10,4)^{5}(8,5)^{5}(6,6)^{5}(4,7)^{4}(2,8)^{3}(0,9) (17,2)(15,3)(13,4)^{2}(11,5)^{2}(9,6)^{2}(7,7)^{2}(5,8)^{2}(3,9)^{2}(1,10)
20	(19,0)^{3}(17,1)^{4}(15,2)^{5}(13,3)^{6}(11,4)^{6}(9,5)^{6}(7,6)^{6}(5,7)^{5}(3,8)^{4}(1,9)^{2} (20,1)(18,2)(16,3)^{2}(14,4)^{2}(12,5)^{3}(10,6)^{2}(8,7)^{3}(6,8)^{2}(4,9)^{3}(2,10)(0,11)
21	(20,0)^{4}(18,1)^{5}(16,2)^{6}(14,3)^{7}(12,4)^{7}(10,5)^{7}(8,6)^{7}(6,7)^{6}(4,8)^{5}(2,9)^{3}(0,10) (21,1)(19,2)^{2}(17,3)^{2}(15,4)^{3}(13,5)^{3}(11,6)^{3}(9,7)^{3}(7,8)^{3}(5,9)^{3}(3,10)^{2}(1,11)
···	···
N odd	(N-1,0)(N-3,1)(N-5,2)···(2,(N-3)/2)(0,(N-1)/2) (N,1)(N-2,2)(N-4,3)···(3,(N-1)/2)(1,(N+1)/2)
N even	(N-1,0)(N-3,1)(N-5,2)···(3,N/2-2)(1,N/2-1) (N,1)(N-2,2)(N-4,3)···(2,N/2)(0,N/2+1)

5. Summary

I have extended the calculational method of the norm kernel on the harmonic oscillator basis to three-cluster systems including a SU(3) non-scalar cluster. Applying it to the ^{39}K+α+α and ^{15}N+α+α three-cluster systems, eigenvalues and eigenstates of the norm kernels have been calculated. They are characterized by SU(3)-classification with the same multiplicity. It is found that the multiplicity of the same (λ, μ)-states increases notably according as N becomes larger and the anti-symmetrizing effect of the three cluster system is still important at large N states. It is necessary to study the structures of ^{47}V and ^{23}Na nuclei using the results of the present calculations.

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(Received May 7, 2013)