Third quantization of $f(R)$-type gravity

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Abstract

We examine the third quantization of $f(R)$-type gravity, based on its effective Lagrangian in the case of a flat Friedmann-Lemaitre-Robertson-Walker metric. Starting from the effective Lagrangian, we make a suitable change of variable and the second quantization, and we obtain the Wheeler-DeWitt equation. The third quantization of this theory is considered. The uncertainty relation of the universe is investigated in the example of $f(R)$-type gravity, where $f(R) = R^2$. It is shown that, at late times namely the scale factor of the universe is large, the spacetime does not contradict to become classical, and, at early times namely the scale factor of the universe is small, the quantum effects dominate.

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1 Introduction

Since the discovery of the accelerated expansion of the universe [1][2][3], much attention has been attracted to the generalized gravity theories of the $f(R)$-type[4][5][6]. Before the discovery, such theories have been interested in because of its theoretical advantages: The theory of the graviton is renormalizable [7][8]. It seems to be possible to avoid the initial singularity of the universe predicted by the theorem by Hawking [9][10]. And inflationary model without inflaton field is possible [11].

On the other hand quantum cosmology is a quantum theory of the universe as a whole, and this is described by the Wheeler-DeWitt equation (WDW eq.), which is a differential equation on the wave function of the universe [12]. However, it is well known that, in general, WDW eq. has the problem that the probabilistic interpretation is difficult as in the case of the Klein-Gordon equation. One of the proposed ideas to solve this problem is the third quantization in analogy with the quantum field theory [13][14][15][16][17] [18][19][20][21][22] [23][24][25][26] [27][28][29]. Then the third-quantized universe theory describes a system of many universes. Third quantization is useful to describe bifurcating universes and merging universes, if an interacting term is introduced in the Lagrangian for the third quantization.

The quantum cosmology of the $f(R)$-type gravity using WDW eq. has already been studied [30]. As noted above the third quantized version is desirable, and the third quantization of $f(R)$-type gravity was also investigated in Ref.[27]. However, in it black holes were studied but cosmology was not treated. So in this work we examine the third quantization of the $f(R)$-type gravity, using explicit form of the action which yields WDW eq. of $f(R)$-type gravity, and we investigate the Heisenberg uncertainty relation of the universe.

We start from the effective theory of the $f(R)$-type gravity in a flat Friedmann-Lemaitre-Robertson-Walker metric. Then a suitable change of variable is performed and WDW eq. is written down. Quantizing this model once more, we obtain the third-quantized theory of $f(R)$-type gravity. The Heisenberg uncertainty relation is investigated in a feasible model of the $f(R)$-type gravity, where $f(R) = R^2$. It will be shown that, at late times namely the scale factor of the universe is large, the spacetime does not contradict to become classical, and, at early times namely the scale factor of the universe is small, the quantum effects dominate.

In section 2, the effective theory of $f(R)$-type gravity in the case of a flat Friedmann-Lemaitre-Robertson-Walker metric is summarized . In section 3, the third quantization of this theory is considered. In section 4, the uncertainty relation is studied in the example of $f(R)$-type gravity, where $f(R) = R^2$. Summary is given in section 5.

2 Generalized gravity of $f(R)$-type

Generalized gravity of $f(R)$-type is one of the higher curvature gravity in which the action is given by

$$S = \int d^4x \mathcal{L} = \int d^4x \sqrt{-g} f(R). \tag{2.1}$$

The spacetime is taken to be 4-dimensional. Here $g \equiv \det g_{\mu\nu}$ and R is the scalar curvature. Field equations are derived by the variational principle as follows

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu}\Box f'(R) = 0, \tag{2.2}$$

where $f'(R) = \dfrac{df(R)}{dR}$, ∇_μ is the covariant derivative with respect to $g_{\mu\nu}$ and $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$.

Let us consider the next action

$$S = \int d^4x \sqrt{-g}\big[f(\phi) + f'(\phi)(R - \phi)\big], \tag{2.3}$$

where $f''(\phi) \neq 0$. It is well known that the field equations of this action are Eq.(2.2) in which $f(R)$ is replaced with $f(\phi)$ and the following equation

$$R = \phi. \tag{2.4}$$

If we substitute Eq.(2.4) into Eq.(2.3), we formally obtain Eq.(2.1).

In order to make things simple, let us consider the case of a flat Friedmann-Lemaitre-Robertson-Walker metric [31],

$$ds^2 = -dt^2 + a^2(t) \sum_{k=1}^{3} (dx^k)^2. \tag{2.5}$$

Then the scalar curvature is written as

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right), \tag{2.6}$$

with $\dot{a} = \dfrac{da}{dt}$. Substituting Eqs.(2.5),(2.6) into Eq.(2.2), we obtain

$$H^2 = \frac{1}{3f'(R)}\left[\frac{1}{2}\Big(Rf'(R) - f(R)\Big) - 3H\dot{R}f''(R)\right], \tag{2.7}$$

$$2\dot{H} + 3H^2 = -\frac{1}{f'(R)}\left[f'''(R)\dot{R}^2 + 2H\dot{R}f''(R) + \ddot{R}f''(R) + \frac{1}{2}\Big(f(R) - Rf'(R)\Big)\right], \tag{2.8}$$

where $H = \dfrac{\dot{a}}{a}$.

Using Eqs.(2.5),(2.6), we can straightforwardly transform Eq.(2.3) to

$$S_{eff} = \int d^4x\Big[a^3 f(\phi) - 6f''(\phi)\dot{\phi}a^2\dot{a} - 6f'(\phi)a\dot{a}^2 - f'(\phi)\phi a^3\Big] = \int d^4x\mathcal{L}_{eff}, \tag{2.9}$$

where the partial integration has been applied to the term containing $\ddot{a}$ [31][30]. The classical equations of motion can be derived from the following equations

$$\frac{\partial\mathcal{L}_{eff}}{\partial a} - \frac{d}{dt}\frac{\partial\mathcal{L}_{eff}}{\partial\dot{a}} = 0, \tag{2.10}$$

$$\frac{\partial\mathcal{L}_{eff}}{\partial\phi} - \frac{d}{dt}\frac{\partial\mathcal{L}_{eff}}{\partial\dot{\phi}} = 0. \tag{2.11}$$

Eqs.(2.6),(2.11) give

$$\phi = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = R. \tag{2.12}$$

We can obtain Eq.(2.8) from Eqs.(2.10),(2.12).

The canonical momenta for a and ϕ are defined as

$$p_a = \frac{\partial\mathcal{L}_{eff}}{\partial\dot{a}} = -12a\dot{a}f'(\phi) - 6a^2f''(\phi)\dot{\phi} = -6a^2(2Hf'(\phi) + f''(\phi)\dot{\phi}), \tag{2.13}$$

$$p_\phi = \frac{\partial\mathcal{L}_{eff}}{\partial\dot{\phi}} = -6a^2\dot{a}f''(\phi) = -6a^3Hf''(\phi). \tag{2.14}$$

So the Hamiltonian is written as

$$\begin{aligned}\mathcal{H}_{eff} &\equiv p_a\dot{a} + p_\phi\dot{\phi} - \mathcal{L}_{eff},\\ &= -\frac{p_a p_\phi}{6a^2f''(\phi)} + \frac{f'(\phi)p_\phi^2}{6a^3f''(\phi)^2} + a^3f'(\phi)\phi - a^3f(\phi).\end{aligned} \tag{2.15}$$

The time reparametrization invariance means

$$\mathcal{H}_{eff} = 0. \tag{2.16}$$

This equation gives Eq.(2.7). Therefore $\mathcal{L}_{eff}$ can be regarded as the effective Lagrangian for Eq.(2.1) when the metric is given by Eq.(2.5).

3 Third quantization

Eqs.(2.15),(2.16) lead to WDW eq.[30] whose kinematic terms are rather complicated, and it is difficult to obtain the action for the third quantization. Therefore, in this section we first make a change of a variable to make the kinematical terms simpler, and then we derive WDW eq. Next we write down the action for the third quantization which yields WDW eq. as the field equation. Then we carry out the third quantization.

Now let us make the change of a variable as follows:

$$\varphi = \varphi(\phi) = \ln f'(\phi), \qquad f'(\phi) = e^{\varphi}, \qquad \phi = f'^{-1}(e^{\varphi}). \tag{3.1}$$

Then the Lagrangian and canonical momenta become

$$\mathcal{L}_{eff} = -6a\dot{a}^2 e^{\varphi} - 6a^2\dot{a}e^{\varphi}\dot{\varphi} - a^3 f'^{-1}(e^{\varphi})e^{\varphi} + a^3 f\big(f'^{-1}(e^{\varphi})\big), \tag{3.2}$$

$$p_a = \frac{\partial \mathcal{L}_{eff}}{\partial \dot{a}} = -12a\dot{a}e^{\varphi} - 6a^2 e^{\varphi}\dot{\varphi}, \tag{3.3}$$

$$p_{\varphi} = \frac{\partial \mathcal{L}_{eff}}{\partial \dot{\varphi}} = -6a^2\dot{a}e^{\varphi}. \tag{3.4}$$

So the Hamiltonian can be written as

$$\begin{aligned} \mathcal{H}_{eff} &= p_a\dot{a} + p_{\varphi}\dot{\varphi} - \mathcal{L}_{eff}, \\ &= \frac{p_{\varphi}^2}{6a^3 e^{\varphi}} - \frac{p_a p_{\varphi}}{6a^2 e^{\varphi}} + a^3 f'^{-1}(e^{\varphi})e^{\varphi} - a^3 f\big(f'^{-1}(e^{\varphi})\big). \end{aligned} \tag{3.5}$$

If we multiply $6a^2e^{\varphi}$ to Eq.(3.5), the Hamiltonian constraint becomes

$$\frac{p_{\varphi}^2}{a} - p_a p_{\varphi} + 6a^5 f'^{-1}(e^{\varphi})e^{2\varphi} - 6a^5 e^{\varphi} f\big(f'^{-1}(e^{\varphi})\big) = 0. \tag{3.6}$$

Substituting

$$p_a \to -i\frac{\partial}{\partial a}, \qquad p_{\varphi} \to -i\frac{\partial}{\partial \varphi}, \tag{3.7}$$

we obtain WDW eq.

$$-\frac{1}{a}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial a\partial\varphi} + 6a^5 f'^{-1}(e^{\varphi})e^{2\varphi}\psi - 6a^5 e^{\varphi} f\big(f'^{-1}(e^{\varphi})\big)\psi = 0. \tag{3.8}$$

Here ψ is the wave function of the universe.

Now let us comment on the possibility of tachyonic states in this WDW eq. In order to examine Eq.(3.8) in the Klein-Gordon form, we make change of variables as

$$\tau = a + \varphi + \ln a, \quad \sigma = a - \varphi - \ln a. \tag{3.9}$$

Then we obtain

$$\frac{\partial^2\psi}{\partial\tau^2} - \frac{\partial^2\psi}{\partial\sigma^2} + U\psi = 0, \tag{3.10}$$

where

$$\begin{aligned} U &= 6a^5 f'^{-1}(e^{\varphi})e^{2\varphi} - 6a^5 e^{\varphi} f\big(f'^{-1}(e^{\varphi})\big), \\ &= 6\left(\tfrac{\tau+\sigma}{2}\right)^3 f'^{-1}\!\left(\tfrac{2}{\tau+\sigma}e^{\frac{\tau-\sigma}{2}}\right)e^{\tau-\sigma} - 6\left(\tfrac{\tau+\sigma}{2}\right)^4 f\left(f'^{-1}\!\left(\tfrac{2}{\tau+\sigma}e^{\frac{\tau-\sigma}{2}}\right)\right)e^{\frac{\tau-\sigma}{2}}. \end{aligned} \tag{3.11}$$

From Eqs.(3.9) we notice that τ can be considered as the time variable, since τ is a monotonic increasing function of the scale factor, a . Therefore, in order to avoid tachyonic states, $U \geq 0$ is required, since U is the square of the effective mass [21]. The condition $U \geq 0$ means

$$f'^{-1}(e^{\varphi})e^{\varphi} \geq f\big(f'^{-1}(e^{\varphi})\big) \tag{3.12}$$

in the original variables. Notice that this condition is satisfied, for example, $f(R) = R^2$.

The action for the third quantization to yield WDW eq.(3.8) can be written as

$$\begin{aligned} S_{3Q} &= \int da d\varphi \frac{1}{2}\left[\frac{1}{a}\left(\frac{\partial \psi}{\partial \varphi}\right)^2 - \frac{\partial \psi}{\partial a}\frac{\partial \psi}{\partial \varphi} + 6a^5 f'^{-1}(e^{\varphi})e^{2\varphi}\psi^2 - 6a^5 e^{\varphi} f\big(f'^{-1}(e^{\varphi})\big)\psi^2\right], \\ &= \int da d\varphi \mathcal{L}_{3Q}. \end{aligned} \tag{3.13}$$

If we consider a to be the time coordinate from now on, the canonical momentum which is conjugate to ψ is written as

$$p_\psi = \frac{\partial \mathcal{L}_{3Q}}{\partial(\partial\psi/\partial a)} = -\frac{1}{2}\frac{\partial \psi}{\partial \varphi}. \tag{3.14}$$

The Hamiltonian is given by

$$\begin{aligned} \mathcal{H}_{3Q} &= p_\psi \frac{\partial \psi}{\partial a} - \mathcal{L}_{3Q}, \\ &= -\tfrac{2}{a}p_\psi^2 - 3a^5 f'^{-1}(e^{\varphi})e^{2\varphi}\psi^2 + 3a^5 e^{\varphi} f\big(f'^{-1}(e^{\varphi})\big)\psi^2. \end{aligned} \tag{3.15}$$

In order to third quantize this system we impose the equal time commutation relation

$$[\hat{\psi}(a,\varphi), \hat{p}_\psi(a,\varphi')] = i\delta(\varphi - \varphi'). \tag{3.16}$$

We use the Schrödinger picture, so we take the operator $\hat{\psi}(a,\varphi)$ as the time independent c-number field $\psi(\varphi)$, and we substitute the momentum operator as

$$\hat{p}_\psi(a,\varphi) \to -i\frac{\partial}{\partial \psi(\varphi)}. \tag{3.17}$$

Then we obtain the Schrödinger equation

$$i\frac{\partial \Psi}{\partial a} = \hat{\mathcal{H}}_{3Q}\Psi, \tag{3.18}$$

that is

$$i\frac{\partial \Psi}{\partial a} = \left[\frac{2}{a}\left(\frac{\partial}{\partial \psi(\varphi)}\right)^2 - 3a^5 f'^{-1}(e^{\varphi})e^{2\varphi}\psi^2(\varphi) + 3a^5 e^{\varphi} f\big(f'^{-1}(e^{\varphi})\big)\psi^2(\varphi)\right]\Psi, \tag{3.19}$$

where Ψ is the third quantized wave function of universes.

4 Uncertainty relation

In order to estimate the uncertainty, we assume the Gaussian ansatz for the solution to the Schrödinger equation (3.18) as is often done

$$\Psi(a,\varphi,\psi(\varphi)) = C\exp\left\{-\frac{1}{2}A(a,\varphi)[\psi(\varphi)-\eta(a,\varphi)]^2 + iB(a,\varphi)[\psi(\varphi)-\eta(a,\varphi)]\right\}, \tag{4.1}$$

where $A(a,\varphi) = D(a,\varphi)+iI(a,\varphi)$ [32][20][22][26]. The real functions $D(a,\varphi), I(a,\varphi), B(a,\varphi)$ and $\eta(a,\varphi)$ should be determined from Eq.(3.19). C is the normalization of the wave function. The inner product of two functions Ψ_1 and Ψ_2 is defined as

$$\langle\Psi_1,\Psi_2\rangle = \int d\psi(\varphi)\Psi_1^*(a,\varphi,\psi(\varphi))\Psi_2(a,\varphi,\psi(\varphi)) \tag{4.2}$$

Let us calculate Heisenberg's uncertainty relation. The dispersion of $\psi(\varphi)$ is defined as $(\Delta\psi(\varphi))^2 \equiv \langle\psi^2(\varphi)\rangle - \langle\psi(\varphi)\rangle^2$. From Eqs.(4.1),(4.2) we have

$$\langle\psi^2(\varphi)\rangle = \frac{1}{2D(a,\varphi)} + \eta^2(a,\varphi), \qquad \langle\psi(\varphi)\rangle = \eta(a,\varphi), \tag{4.3}$$

then

$$(\Delta\psi(\varphi))^2 = \frac{1}{2D(a,\varphi)}. \tag{4.4}$$

Similarly the dispersion of $p_\psi(\varphi)$ is defined as $(\Delta p_\psi(\varphi))^2 \equiv \langle p_\psi^2(\varphi)\rangle - \langle p_\psi(\varphi)\rangle^2$. We obtain

$$\langle p_\psi^2(\varphi)\rangle = \frac{D(a,\varphi)}{2} + \frac{I^2(a,\varphi)}{2D(a,\varphi)} + B^2(a,\varphi), \qquad \langle p_\psi(\varphi)\rangle = B(a,\varphi), \tag{4.5}$$

then we have

$$(\Delta p_\psi(\varphi))^2 = \frac{D(a,\varphi)}{2} + \frac{I^2(a,\varphi)}{2D(a,\varphi)}. \tag{4.6}$$

Therefore the uncertainty relation can be written as

$$(\Delta\psi(\varphi))^2(\Delta p_\psi(\varphi))^2 = \frac{1}{4}\left(1+\frac{I^2(a,\varphi)}{D^2(a,\varphi)}\right). \tag{4.7}$$

Note that to evaluate (4.7), only $A(a,\varphi)$ is necessary. However, it is difficult to solve the equation for $A(a,\varphi)$ for general $f(R)$, so let us take a feasible example such as

$$f(R) = R^2. \tag{4.8}$$

In this case

$$f'(R) = 2R = e^\varphi, \qquad f'^{-1}(e^\varphi) = R = \frac{e^\varphi}{2}, \qquad f\left(f'^{-1}(e^\varphi)\right) = \frac{e^{2\varphi}}{4}. \tag{4.9}$$

The Schrödinger equation (3.19) becomes

$$i\frac{\partial\Psi}{\partial a} = \left[\frac{2}{a}\left(\frac{\partial}{\partial\psi(\varphi)}\right)^2 - \frac{3}{4}a^5e^{3\varphi}\psi^2(\varphi)\right]\Psi. \tag{4.10}$$

Substituting the ansatz (4.1) into Eq.(4.10), we obtain

$$-\frac{i}{2}\frac{\partial A(a,\varphi)}{\partial a}=\frac{2}{a}A^2(a,\varphi)-\frac{3}{4}a^5e^{3\varphi}. \tag{4.11}$$

Writing

$$\ln a=\frac{\alpha}{6}, \tag{4.12}$$

we have

$$-3i\frac{\partial A(\alpha,\varphi)}{\partial\alpha}=2A^2(\alpha,\varphi)-\frac{3}{4}e^{\alpha}e^{3\varphi}. \tag{4.13}$$

In order to solve this equation, let us write

$$A(\alpha,\varphi)=\frac{3i}{2}\frac{\partial\ln u(\alpha,\varphi)}{\partial\alpha}, \tag{4.14}$$

where $u(\alpha,\varphi)$ is a suitable function. Then $u(\alpha,\varphi)$ must satisfy the equation,

$$\frac{\partial^2 u(\alpha,\varphi)}{\partial\alpha^2}+\frac{1}{6}e^{3\varphi}e^{\alpha}u(\alpha,\varphi)=0. \tag{4.15}$$

Now we write

$$z=2\sqrt{\frac{e^{3\varphi}e^{\alpha}}{6}}, \tag{4.16}$$

and we have

$$\frac{\partial^2 u(z,\varphi)}{\partial z^2}+\frac{1}{z}\frac{\partial u(z,\varphi)}{\partial z}+u(z,\varphi)=0. \tag{4.17}$$

As this equation can be regarded as the ordinary differential equation with respect to z under the assumption that φ is fixed other than in z, this is the case when $\nu=0$ in the following Bessel's equation

$$\frac{d^2u(z)}{dz^2}+\frac{1}{z}\frac{du(z)}{dz}+\left(1-\frac{\nu^2}{z^2}\right)u(z)=0. \tag{4.18}$$

Therefore we have the solution

$$u(z,\varphi)=c_J(\varphi)J_0(z)+c_Y(\varphi)Y_0(z), \tag{4.19}$$

where $J_0,\ Y_0$ are the Bessel functions of order 0 and $c_J,\ c_Y$ are arbitrary complex functions of φ.

From Eqs.(4.9),(4.12),(4.14),(4.16),(4.19), we can obtain $z=\frac{4}{\sqrt{3}}R^{\frac{3}{2}}a^3$, $\varphi=\ln(2R)$ and

$$A(z,\varphi)=-i\frac{3z}{4}\frac{c_J(\varphi)J_1(z)+c_Y(\varphi)Y_1(z)}{c_J(\varphi)J_0(z)+c_Y(\varphi)Y_0(z)}, \tag{4.20}$$

where we have used $J_0'(z)=-J_1(z),\ Y_0'(z)=-Y_1(z)$ [33].

Since $A(z,\varphi)=D(z,\varphi)+iI(z,\varphi)$, we have

$$D(z,\varphi)=-\frac{3i}{4\pi|c_J(\varphi)J_0(z)+c_Y(\varphi)Y_0(z)|^2}[c_J(\varphi)c_Y^*(\varphi)-c_J^*(\varphi)c_Y(\varphi)], \tag{4.21}$$

where we used $J_0(z)Y_1(z) - J_1(z)Y_0(z) = -\dfrac{2}{\pi z}$ [33], and

$$\begin{aligned} I(z,\varphi) = & -\frac{3z}{8|c_J(\varphi)J_0(z)+c_Y(\varphi)Y_0(z)|^2}\times \\ & \Big[2|c_J(\varphi)|^2 J_0(z)J_1(z) + 2|c_Y(\varphi)|^2 Y_0(z)Y_1(z) \\ & +(c_J(\varphi)c_Y^*(\varphi)+c_J^*(\varphi)c_Y(\varphi))(J_0(z)Y_1(z)+J_1(z)Y_0(z))\Big]. \end{aligned} \tag{4.22}$$

So if we assume $c_J(\varphi)c_Y^*(\varphi) - c_J^*(\varphi)c_Y(\varphi) \neq 0$ (Note that in this case both of $c_J(\varphi), c_Y(\varphi)$ are nonzero.), we obtain

$$\begin{aligned} \frac{I^2(z,\varphi)}{D^2(z,\varphi)} = & -\frac{\pi^2 z^2}{4[c_J(\varphi)c_Y^*(\varphi)-c_J^*(\varphi)c_Y(\varphi)]^2}\times \\ & \Big[2|c_J(\varphi)|^2 J_0(z)J_1(z) + 2|c_Y(\varphi)|^2 Y_0(z)Y_1(z) \\ & +(c_J(\varphi)c_Y^*(\varphi)+c_J^*(\varphi)c_Y(\varphi))(J_0(z)Y_1(z)+J_1(z)Y_0(z))\Big]^2. \end{aligned} \tag{4.23}$$

At late times namely $a \to \infty$ i.e. $z \to \infty$,

$$\begin{aligned} J_0(z) &\sim \sqrt{\tfrac{2}{\pi z}}\cos(z-\tfrac{\pi}{4}), \qquad J_1(z) \sim \sqrt{\tfrac{2}{\pi z}}\sin(z-\tfrac{\pi}{4}), \\ Y_0(z) &\sim \sqrt{\tfrac{2}{\pi z}}\sin(z-\tfrac{\pi}{4}), \qquad Y_1(z) \sim -\sqrt{\tfrac{2}{\pi z}}\cos(z-\tfrac{\pi}{4}), \end{aligned}$$

[33] we have

$$\begin{aligned} \frac{I^2(z,\varphi)}{D^2(z,\varphi)} & \sim -\frac{[(|c_J(\varphi)|^2-|c_Y(\varphi)|^2)\cos(2z)+(c_J(\varphi)c_Y^*(\varphi)+c_J^*(\varphi)c_Y(\varphi))\sin(2z)]^2}{[c_J(\varphi)c_Y^*(\varphi)-c_J^*(\varphi)c_Y(\varphi)]^2} \\ & \sim O(1). \end{aligned} \tag{4.24}$$

This and Eq.(4.7) mean that at late times namely $a \to \infty$, it is plausible that the spacetime becomes classical.

On the other hand at early times namely $a \to 0$ i.e. $z \to 0$,

$$\begin{aligned} J_0(z) &\sim 1-\tfrac{z^2}{4}, \qquad J_1(z) \sim \tfrac{z}{2}, \\ Y_0(z) &\sim \tfrac{2}{\pi}\Big(\ln\tfrac{z}{2}+\gamma\Big), \qquad Y_1(z) \sim -\tfrac{2}{\pi z}, \end{aligned}$$

[33] we obtain

$$\frac{I^2(z,\varphi)}{D^2(z,\varphi)} \sim -\frac{16|c_Y(\varphi)|^4}{\pi^2[c_J(\varphi)c_Y^*(\varphi)-c_J^*(\varphi)c_Y(\varphi)]^2}\Big(\ln\frac{z}{2}+\gamma\Big)^2 \sim \infty, \tag{4.25}$$

where γ is a constant. This and Eq.(4.7) mean that the fluctuation of the third quantized universe field becomes large at early times namely $a \to 0$. Therefore the quantum effects dominate for the small values of the universe radius.

5 Summary

In this work the third quantization of the $f(R)$-type gravity is investigated, when the metric is a flat Friedmann-Lemaitre-Robertson-Walker one. The Heisenberg uncertainty relation is investigated in a feasible model of the $f(R)$-type gravity, that is $f(R) = R^2$. It has been shown in this model that, at late times namely the scale factor of the universe is large, the spacetime does not contradict to become classical, and, at early times namely the scale factor of the universe is small, the quantum effects dominate. This result is similar to Ref.[21][22][26] but is not similar to Ref.[20], where it was shown that quantum effects dominate also when the scale factor is large. However, as pointed out in Ref.[22] this era corresponds to the classically forbidden region in the model[20].

As a future work, it will be interesting to investigate a more realistic model such as $f(R) = R + cR^2$, where c is a constant. Though our formulation started from the effective action (2.3), it will be also interesting to quantize (2.1) directly as in Ref.[27], since the quantization of the $f(R)$ -type gravity is not unique.

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