On The Scale Of Correlation of Stochastic Fields

ON THE CORRELATION SCALE OF STOCHASTIC FIELDS

T. Harada[1]

and

M. Shinozuka[2]

1986

1. Visiting Research Scientist, Department of Civil Engineering and Engineering Mechanics, Columbia University; Associate Professor, Faculty of Engineering, Miyazaki University, Miyazaki, Japan.

2. Renwick Professor of Civil Engineering, Department of Civil Engineering and Engineering Mechanics, Columbia University, New York, NY.

1. INTRODUCTION

The following are well known in the second-order analysis of stochastic process theory: (1) a homogeneous stochastic process is usually characterized by its mean value and correlation function, (2) the correlation function represents the variance and correlation structure of the process, (3) the correlation function is related to the power spectral density function by means of the Wiener-Khintchine transform pair, and (4) if the process is Gaussian, then all its characteristics are known only from its mean value and correlation function or power spectral density function. Therefore, when stochastic process theory is applied to the analysis of observed field data, a set of these observations is interpreted as a realization of a homogeneous stochastic process. Then, the mean value and correlation function or power spectral density function are usually estimated following routes 1 and 2 shown schematically in Fig. 1. Finally the resulting correlation function and power spectral density function are in general summarized in analytical terms.

In the above procedures usually encountered in practical field data analyses, the last step of modeling is, of course, based on not onlythe observed data, but also physical understanding of the phenomena and engineering judgment. Hence, the modeling task cannot be successfully achieved without understanding the phenomena indicated by the observed data and without taking into account the accuracy required for analyses.

In addition to the correlation function, however, if simple statistics (correlation scale) could be defined that are able to represent the correlation structure of the stochastic processes and also can be directly estimated from a set of observed field data without the correlation function or the power spectral density function, these statistics for correlation could provide quite useful information for capturing the essential phenomena indicated by the observed data and eventually in the modeling of its stochastic process. Consequently, it is asserted that three statistics (the mean, variance and correlation study) could be used as the fundamental parameters to approximately characterize stochastic processes. In fact, as briefly described in Section 1.1, instead of the correlation function, the correlation scales summarized in Table 1 have been successfully used as measures of the correlation structure of stochastic processes in studies of the turbulence, signal analysis and stochastic response of mechanical systems to dynamic loading. However, since these correlation scales are defined through the correlation function, they cannot be evaluated before knowing the correlation function. From the viewpoint of the statistical analysis of stochastic process data, it is desirble to estimate the correlation scale directly from observed data without going through the correlation function. Hence the definitions for the correlation scales indicated in Table 1 are quite useless from the point of view of the statistical analysis of stochastic process data.

In this study, two new definitions for the correlation studies A and C in Table 1 are discussed which are suitable for the statistical analysis of observed data in the sense described above. Hence, the problem dealt with in this study is to develop a practical procedure for estimating correlation scales directly from observed data iwthout going through the correlation function (route 4 in Fig. 1). To do this, the variance behavior of an averaging process previously studied by Panchev (1971), Bendat and Piersol (1971) and Vanmarcke (1983) is analyzed in a systematic way. (The procedure used in this study is especially similar to that used by Vanmarcke.) However, the results and their interpretation are quite different from those of previous studies, and the two different definition for correlation scales are reinterpreted in a consistent way from the viewpoint of the statistical analysis of stochastic process data. Consequently, a practical procedure utilizing a graphical method as occasion demands, is presented to estimate the correlation studies directly from observed data without using the correlation function. The procedure for one-dimensional stochastic process data is also extended to the two-dimensional case and the significance of the correlation scales for two-dimensional stochastic processes is briefly discussed using numerically generated two-dimensional stochastic fields. In the final chapter, some new application examples of correlation scales are briefly presented.

1.1 Brief Historical Note on Correlation Scales

Table 1 summarizes the definitions of correlation scales in the literature available. In the study of turbulence, G.I. Taylor (1935) first proposed a measure of the correlation scale to obtain low variance estimates of the mean value of fluctuating velocities. The ratio of a finite sampling interval to the correlation (A in Table 1) is used as the equivalent number of independent observations from stochastic process data. G.K. Batchelor (1953), V.I. Tatarski (1961), and A.S. Monin and A.M. Yaglam (1965) also used the same measure proposed by G.I. Taylor (1921) in their studies of isotropic turbulence. In the study of random signal analysis (S. Panchev, 1971, J.S. Bendat and A.G. Piersol, 1971), the correlation scale A in Table 1 was also used for the condition of ergodicity with respect to the mean value. R.L. Stratonovitch (1967) used the other definition (B) as indicated in Table 1 in the discussion of the condition of ergodicity with respect to the mean value by considering the averaging process. Correlation scale C in Table 1 was proposed to represent the inner scale of turbulence (V.I. Tatarski 1961, A.S. Monin and A.M. Yaglom 1965, J.L. Lumley, 1970). In the study of stochastic respons of mechanical systems to dynamic loading (Y-K. Lin et al, 1979), the other definition of correlation scale D in Table 1 was used. This correlation scale is proposed in such a way that if the correlation scale (time) of a dynamic loading is much smaller larger than the relaxation time of the mechanical system, then the response can be approximated by a Markov process. Thus, many convenient mathematical properties related to Markov processes can be used to solve the system response to dynamic loading (R.L. Stratonovitch, 1967; Y-K. Lin, 1979; W-F. Wu, 1985).

Recently, Vanmarcke (1983) reinterpreted the correlation scale in A in Table 1 (scale of fluctuation) from the viewpoint of the analysis of the variance of averaging processes in a manner similar to the discussion of Stratonovitch (1963) and S. Pancheve (1971), and rpesented many applications in civil and mechanical engineering problems. Harada and Shinozuka (1985) recently proposed correlation scale C in Table 1 in their analysis of the spatial variations of seismic ground motions by considering the variance of difference processes.

In conclusion, previous definitions for corelation scales were all based on the correlation function or the power spectral density function and tend to be vague in why they are defined as shown in Table 1, except the studies of Vanmarcke, and Harada and Shinozuka. Thus, to obtain the correlation scale, the correlation funciton or the power spectral density function has to be given first. This kind of definition is not useful rom the viewpoint of the statistical analysis of observed data because it is desirable to estimate the correlation scale directly from the observed data without using the correlation function.

2. VARIANCE OF AVERAGING PROCESS AND DIFFERENCE PROCESS

Since any continuous parameter homogeneous stochastic process with mean m and variance σ_{ff}^2 can be expressed as the sum of its mean m and homogeneous stochastic process f(x) with zero mean and variance σ_{ff}^2, we consider a homogeneous stochastic process f(x) with zero mean and variance σ_{ff}^2 in the analysis that follows.

For a homogeneous stochastic process f(x), a family of the averaging process $f_D(x)$ may be defined such that

$$f_D(x) = \frac{1}{D} \int_{x-\frac{D}{2}}^{x+\frac{D}{2}} f(y)dy \tag{2.1}$$

Introducing the following indefinite integral F(x) of f(x)

$$F(x) = \int_{-\infty}^{x} f(y)dy \qquad \text{or} \qquad \frac{dF(x)}{dx} = f(x) \tag{2.2}$$

Equation 1 is also written as

$$f_D(x) = \frac{1}{D} F_D(x) \tag{2.3}$$

where

$$F_D(x) = F\left(x+\frac{D}{2}\right) - f\left(x-\frac{D}{2}\right) \tag{2.4}$$

The function $F_D(x)$ is the finite difference process of F(x). In Eqs. 1 and 3, the averaging process $f_D(x)$ and difference process $F_D(x)$ are always homogeneous since the original process f(x) is homogeneous. However, the indefinite integral process F(x) is not always homogeneous. The condition for the homogeneity of F(x) is very closely related to the behavior of the power spectral density function $S_{ff}(\kappa)$ of f(x) at origin $\kappa = 0$. If F(x) is homogeneous, the power spectral density function $S_{FF}(\kappa)$ of F(x) is well known to be expressed due to Eq. 2 as

$$S_{FF}(\kappa) = \frac{S_{ff}(\kappa)}{\kappa^2} \tag{2.5}$$

As is also well known, $S_{ff}(\kappa)$ is in turn related to the correlation function $R_{ff}(\xi)$ through the Wiener-Khintchine transform pair:

$$S_{ff}(\kappa) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{ff}(\xi)e^{-i\kappa\xi}\, d\xi \tag{2.6a}$$

$$R_{ff}(\xi) = \int_{-\infty}^{\infty} S_{ff}(\kappa)e^{i\kappa\xi}\, d\kappa \tag{2.6b}$$

Accounting for the symmetry of $R_{ff}(\xi)$ with respect to the origin ($R_{ff}(\xi) = R_{ff}(-\xi)$), the Wiener-Khintchine transform pair is also given as

$$S_{ff}(\kappa) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{ff}(\xi)\cos\kappa\xi d\xi \tag{2.6c}$$

$$R_{ff}(\xi) = \int_{-\infty}^{\infty} S_{ff}(\kappa)\cos\kappa\xi d\kappa \tag{2.6d}$$

Using the asymptotic expansion of $\cos\kappa\xi$, $S_{ff}(\kappa)$ can be expressed as

$$\begin{aligned} S_{ff}(\kappa) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{ff}(\xi)\Big[\sum_{n=0}^{\infty}(-1)^n \frac{(\kappa\xi)^{2n}}{(2n)!}\Big]d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{ff}(\xi)d\xi - \frac{1}{2\pi}\frac{1}{2!}\kappa^2\int_{-\infty}^{\infty}\xi^2 R_{ff}(\xi)d\xi + \cdots\cdot \end{aligned} \tag{2.7}$$

Then, from Eqs. 2.5 and 2.7, $S_{FF}(\kappa)$ is also expressed as

$$S_{FF}(\kappa) = \frac{1}{2\pi}\cdot\frac{1}{\kappa^2}\int_{-\infty}^{\infty} R_{ff}(\xi)d\xi - \frac{1}{2\pi}\frac{1}{2!}\int_{-\infty}^{\infty}\xi^2 R_{ff}(\xi)d\xi + \cdots\cdot \tag{2.8}$$

Therefore, if $S_{ff}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{ff}(\xi)d\xi \neq 0$, then $S_{FF}(\kappa)$ is singular at the origin. This means that the variance of $F(x)$ becomes infinity and the process $F(x)$ is no longer homogeneous.

It should be lnoted again that the difference process $F_D(x)$ of $F(x)$ given by Eq. 2.4 is always homogeneous even in the case where the process $F(x)$ is non-homogeneous because the averaging process $f_D(x)$ is always homogeneous (see

Eq. 2.1). More rigorous discussion concerning the homogeneity of the integral and difference processes can be4 seen in the following (Cramer and Leadbetter, 1967; Doob, 1953; and Yaglom, 1962, 1973).

Turning to the variance σ_D^2 of the averaging process $f_D(x)$, we first consider the power spectral density function $S_{f_D}(\kappa)$ of $f_D(x)$. $S_{f_D}(\kappa)$ is given as follows:

$$S_{f_D}(\kappa) = \left[\frac{\sin \frac{\kappa D}{2}}{\frac{\kappa D}{2}}\right]^2 S_{ff}(\kappa) \tag{2.9}$$

Equation 2.9 is derived from the following general well-known equations in filtering theory (for example, Papoulis 1984):

$$f_D(x) = \int_{-\infty}^{\infty} f(u)h(x-u)du = \int_{-\infty}^{\infty} f(x-u)h(u)du \tag{2.10a}$$

where h(u) is the impulse response function of a system, and the power spectral density function of the response $f_D(x)$ to the input f(x) is given by

$$S_{f_D}(\kappa) = |H(\kappa)|^2 S_{ff}(\kappa) \tag{2.10b}$$

where $H(\kappa)$ is the transfer function of the system which is related to $h(\kappa)$ such that

$$H(\kappa) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(u)e^{-i\kappa u}\, du \tag{2.10c}$$

Since the averaging process $f_D(x)$ as defined in Eq. 2.1 is identical with Eq. 2.10a having

$$\begin{aligned} h(u) &= \frac{1}{D} && -\frac{D}{2} \leq u \leq \frac{D}{2} \\ &= 0 && \text{otherwise} \end{aligned} \tag{2.11}$$

and hence

$$H(\kappa) = \frac{\sin \frac{kD}{2}}{\frac{kD}{2}} \tag{2.12}$$

Substitution of Eq. 2.12 into Eq. 2.10b yields Eq. 2.9.

The variance σ_D^2 of $f_D(x)$ is given following its definition such that

$$\sigma_D^2 = E\left[f_D^2(x)\right] = R_{f_D}(0) = \int_{-\infty}^{\infty} S_{f_D}(\kappa)d\kappa \tag{2.13}$$

where $E[\cdot]$ is the expectation operator and $R_{f_D}(\xi)$ is the correlation function of $f_D(x)$. Then, substitution of Eq. 2.9 into Eq. 2.13 gives the variance σ_D^2 in terms of the power spectral density function $S_{ff}(\kappa)$ of $f(x)$ as follows:

$$\sigma_D^2 = \int_{-\infty}^{\infty} \left[\frac{\sin \frac{\kappa D}{2}}{\frac{kD}{2}}\right]^2 S_{ff}(\kappa)d\kappa \tag{2.14}$$

Equation 2.14 is also rewritten using the relationship between the basic spectral window and the log window as follows:

$$\left[\frac{\sin \frac{kD}{2}}{\frac{kD}{2}}\right]^2 = \frac{1}{D}\int_{-\infty}^{\infty} \left(1 - \frac{|\xi|}{D}\right)\cos \kappa\xi d\xi \tag{2.15}$$

Then, from Eqs. 2.6d, 2.14 and 2.15,

$$\sigma_D^2 = \frac{1}{D}\int_{-\infty}^{\infty} \left(1 - \frac{|\xi|}{D}\right)R_{ff}(\xi)d\xi \tag{2.16}$$

If $F(x)$ is homogeneous, Eq. 2.14 is also expressed in terms of the correlation function $R_{FF}(\xi)$ of the indefinite integral process $F(x)$ as follows:

$$\begin{aligned}
\sigma_D^2 &= \int_{-\infty}^{\infty} \left[\frac{\sin \frac{\kappa D}{2}}{\frac{\kappa D}{2}}\right]^2 S_{ff}(\kappa)d\kappa \\
&= \left(\frac{2}{D}\right)^2 \int_{-\infty}^{\infty} \frac{S_{ff}(\kappa)}{\kappa^2} \sin^2\left(\frac{\kappa D}{2}\right)d\kappa \\
&= \left(\frac{2}{D}\right)^2 \int_{-\infty}^{\infty} S_{FF}(\kappa)\sin^2\left(\frac{\kappa D}{2}\right)d\kappa
\end{aligned}$$

$$= \left(\frac{2}{D}\right)^2 \int_{-\infty}^{\infty} S_{FF}(\kappa)\left\{\frac{1 - \cos\kappa D}{2}\right\} d\kappa$$

$$= \frac{2}{D^2}\left[R_{FF}(0) - R_{FF}(D)\right] \tag{2.17}$$

Summarizing Eqs. 2.14, 2.16 and 2.17, one has

$$\sigma_D^2 = \int_{-\infty}^{\infty} \left[\frac{\sin\frac{\kappa D}{2}}{\frac{\kappa D}{2}}\right]^2 S_{ff}(\kappa) d\kappa \tag{2.18}$$

$$= \frac{1}{D}\int_{-\infty}^{\infty} \left(1 - \frac{|\xi|}{D}\right) R_{ff}(\xi) d\xi \tag{2.19}$$

and, when $S_{ff}(0) = 0$, and hence the indefinite integral process $F(x)$ is homogeneous,

$$\sigma_D^2 = \frac{2}{D^2}\left[R_{FF}(0) - R_{FF}(D)\right] \tag{2.20}$$

The variance σ_D^2 of the difference process $F_D(x)$ is related to σ_D^2 due to Eq. 2.3 as follows:

$$\sigma_{F_D}^2 = D^2\sigma_D^2 \tag{2.21}$$

The relationships between the processes $f(x)$, $f_D(x)$, $F(x)$ and $F_D(x)$ may also be summarized schematically as shown in Fig. 1. In the case where the power spectral density function at the origin $S_{ff}(0) \neq 0$, the indefinite integral process $F(x)$ of $f(x)$ becomes nonhomogeneous, but otherwise $F(x)$, $F_D(x)$ and $f_D(x)$ are all homogeneouis stochastic processes with variances σ_{FF}^2, $\sigma_{F_D}^2$ and σ_D^2, respectively. Equations 2.18-2.21 and the summary shown schematically in Fig. 1 play a fundamental role in the new interpretation of the definition of the correlation scale which is capable of being estimated directly from the observation data without recourse to the correlation function.

3. DEFINITION AND SIGNIFICANCE OF CORRELATION SCALE

We consider in this chapter the behavior of the variance σ_D^2 of $f_D(x)$ in the two limiting cases where $D \to 0$ and $D \to \infty$, using Eqs. 2.18-2.21. And then, we introduce new definitions for the correlation scales.

In the first case where D becomes zero, using Eq. 2.18 together with the relationship of $\sin(\frac{\kappa D}{2}) = \frac{\kappa D}{2}$ for $D \to 0$, we can easily show that

$$\sigma_D^2 = \int_{-\infty}^{\infty} S_{ff}(\kappa) d\kappa = \sigma_{ff}^2 \qquad \text{as } D \to 0 \tag{3.1}$$

If $S_{ff}(0) = 0$, then the integral process F(x) is homogeneous. Hence we consider Eq. 2.20. The correlation function $R_{FF}(D)$ can be expanded into a Taylor series around D = 0:

$$R_{FF}(D) = R_{FF}(0) + F''_{FF}(0)D + \frac{1}{2!} R''_{FF}(0)D^2 + \cdots\cdot \tag{3.2}$$

where $R'_{FF}(0) = dR_{FF}(\xi)/d\xi \Big|_{\xi=0}$ and similar definitions apply to $R''_{FF}(0)$, etc. Due to the assumption that the original process f(x) is homogeneous, we obtain

$$2\left[R_{FF}(0) - R_{FF}(D)\right] = -R''_{FF}(0)D^2 = \sigma^2 D^2 \qquad \text{as } D \to 0 \tag{3.3}$$

Making use of the apparent wave length L_F of the process F(X),

$$L_F = 2\pi \frac{\sigma_{FF}}{\sigma_{ff}} = 2\pi\sqrt{-\frac{R_{FF}(0)}{R''_{FF}(0)}} = 2\pi\sqrt{\frac{\int_{-\infty}^{\infty} S_{FF}(\kappa)d\kappa}{\int_{-\infty}^{\infty} S_{ff}(\kappa)d\kappa}} \tag{3.4}$$

we introduce the correlation scale L_F^* as

$$L_F^* = \frac{1}{\sqrt{2}\,\pi} L_F \tag{3.5}$$

and

$$\sigma_{FF} = \frac{1}{\sqrt{2}} L_F^* \sigma_{ff} \tag{3.6}$$

Then, it follows from Eqs. 2.21, 3.3 and 3.6, that

$$\sigma_D^2 = \sigma_{FF}^2 = \left(\frac{\sqrt{2}}{L_F^*}\right)\sigma_{FF}^2 \qquad \text{as } D \to 0 \text{ when } S_{ff}(0) = 0 \tag{3.7}$$

In the second case where $D \to \infty$, the second term of the integral in Eq. 2.19 approaches zero (for proof, see, for example, Y-K. Lin, 1967, pp. 57-58). Thus, using Eq. 2.19, the variance σ_D^2 is approximately given as

$$\sigma_D^2 = \frac{1}{D}\int_{-\infty}^{\infty} R_{ff}(\xi)d\xi \qquad \text{as } D \to \infty \tag{3.8}$$

Equation 3.8 is also derived using Eq. 2.18 together with the fact that the only values associated with the wave number κ near zero contribute to the integral in Eq. 2.18 for $D \to \infty$ as follows:

$$\begin{aligned}\sigma_D^2 &= \frac{2}{D}\int_{-\infty}^{\infty} \left[\frac{\sin u}{u}\right]^2 S_{ff}\left(\frac{2u}{D}\right)du \\ &\approx \frac{2}{D} S_{ff}(0)\int_{-\infty}^{\infty}\left[\frac{\sin u}{u}\right]^2 du \\ &= \frac{2\pi}{D} S_{ff}(0) \\ &= \frac{1}{D}\int_{-\infty}^{\infty} R_{ff}(\xi)d\xi\end{aligned} \tag{3.9}$$

In obtaining the last two elements in Eq. 3.9, the Wiener-Khintchine relationship given by Eq. 2.6 and the following definite integral value are used:

$$\int_{-\infty}^{\infty} \left[\frac{\sin u}{u}\right]^2 du = \pi \tag{3.10}$$

When $S_{ff}(0) = 0$ ($F(x)$ is homogeneous), the variance σ_D^2 for $D \to \infty$ can be given, using Eq. 2.20 together with $R_{FF}(D) \to 0$ as $D \to \infty$, such that

$$\sigma_D^2 = \frac{2}{D^2} R_{FF}(0) = \frac{2}{D^2}\sigma_{FF}^2 \tag{3.11}$$

Also using Eq. 3.6, Eq. 3.11 can be expressed such that

$$\sigma_D^2 = \left(\frac{L_F^*}{D}\right)^2 \sigma_{FF}^2 \tag{3.12}$$

Equations 3.8 and 3.9 are identical with those used in the condition of ergodicity with respect to the mean value where the variance σ_D^2 is interpreted as the mean square error of the estimate mean value of f(x) over a finite length D (Panchev, 1971; Bendat and Persol, 1971). In this case, Eqs. 3.8 and 3.9 are interpreted as follows: When the integrals in Eqs. 3.8 and 3.9 are finite-valued, the mean square error of the estimated mean of f(x) approaches zero as $D \to \infty$, providing that the estimate is a "consistent" estimate of the mean value of f(x) (see Section 5.3).

Summarizing Eqs. 2.18, 2.19, 2.20, 3.1, 3.7, 3.11 and 3.12, when $S_{ff}(0) \neq 0$ (Case I):

$$\sigma_D^2 = \int_{-\infty}^{\infty} \left[\frac{\sin\frac{\kappa D}{2}}{\frac{\kappa D}{2}}\right]^2 S_{ff}(\kappa)d\kappa = \frac{1}{D}\int_{-\infty}^{\infty}\left(1 - \frac{|\xi|}{D}\right)R_{ff}(\xi)d\xi \tag{3.13a}$$

$$= \sigma_{ff}^2 \qquad \text{as } D \to 0 \tag{3.13b}$$

$$= \frac{L^*}{D}\,\sigma_{ff}^2 \qquad \text{as } D \to \infty \tag{3.13c}$$

where L* is the correlation scale defined as

$$L^* = \frac{1}{\sigma_{ff}^2}\,2\pi S_{ff}(0) = \frac{1}{\sigma^2}\int_{-\infty}^{\infty} R_{ff}(\xi)d\xi \tag{3.14}$$

when $S_{ff}(0) = 0$, and then F(x) is a homogeneous process with variance σ_F^2 (Case II):

$$\sigma_D^2 = \frac{2}{D^2}\left[R_{FF}(0) - R_{FF}(D)\right] \tag{3.15a}$$

$$= \sigma_{FF}^2 = \left(\frac{\sqrt{2}}{L_F^*}\right)\sigma_F^2 \qquad \text{as } D \to 0 \tag{3.15b}$$

$$= \left(\frac{L_F^*}{D}\right)^2 \sigma_{ff}^2 = \left(\frac{\sqrt{2}}{D}\right)^2 \sigma_{FF}^2 \qquad \text{as } D \to \infty \tag{3.15c}$$

where L_F^* is also the correlation scale defined by Eq. 3.5.

Although the correlation scales L_F^* and L^* defined by Eqs. 3.5 and 3.4 possess the same forms as A and C in Table 1, which are defined by previous investigators, the significance of the correlation scales defined in this study is quite clear: The correlation scale L_F^* defined by Eq. 3.5 is such that when the relative distance D reaches the distance of the correlation scale L_F^*, the variance σ_D^2 becomes $2\sigma_{FF}^2/D^2$ according to Eq. 3.15b. This is the same variance of σ_D^2 when F(x+D) and F(x) become completely uncorrelated as D $\to \infty$ (see Eqs. 3.15a and 3.15c). Also, for the correlation scale L^* defined by Eq. 3.4, a similar consideration can be made using Eq. 3.13 as follows: When the averaging distance D reaches the distance of the correlation scale L^*, the variance of the averaging process becomes σ^2 in accordance with Eq. 3.13c. This is the same variance of the averaging process ehn the original process f(x) may be considered to be a perfectly correlated process, i.e., $R_{ff}(\xi) = \sigma_{ff}^2$ for $D \to 0$ (see Eq. 3.13a and 3.13b).

The definitions of the correlation scales L^* and L_F^* are also interpreted in t6erms of the wave number κ as follows: Since the wave number κ is related to the wave length L such that $\kappa = 2\pi/L$, we may define the spectral scales κ^* and κ_F^* corresponding to the correlation scales (distances) L^* and L_F^*, respectively such that

$$‡^* = \frac{2\pi}{L^*} \qquad \text{and} \qquad \kappa_F^* = \frac{2\pi}{L_F^*} \tag{3.16}$$

Then, Eq. 3.16 can be written using Eqs. 3.4, 3.5 and 3.14 together with the Wiener-Khintchine relationship given by Eq. 2.6 as follows,

$$\kappa^* = \frac{1}{S_{ff}(0)} \int_{-\infty}^{\infty} S_{ff}(\kappa) d\kappa \tag{3.17}$$

and

$$\kappa_F^* = \sqrt{2}\ \pi \sqrt{\frac{\int_{-\infty}^{\infty} S_{F'F'}(\kappa)d\kappa}{\int_{-\infty}^{\infty} S_{FF}(\kappa)d\kappa}} = \sqrt{2}\ \pi\ \kappa_F \tag{3.18}$$

where κ_F is the apparent wave number. The significance of the spectral scales defined by Eqs. 3.16 and 3.17 is illustrated in Fig. 1. It may be observed from Fig. 1 that the spectral scale represents a large wave number above which the power spectral density function may be considered to be zero.

As demonstrated in the numerical example, definitions of the correlation scales given by Eqs. 3.13-3.15 are also useful for estimating the correlation scale from the observed data since the variances σ_{ff}^2 and σ_D^2 or σ_{FF}^2 and $\sigma_{F_D}^2$ can be easily calculated by following their definitions from the observed data.

4. GRAPHICAL REPRESENTATION

A grpahical representation of Eqs. 3.13-3.15 as shown in Fig. 1 may be more useful for estimating the correlation scale L* or L_F^* from a set of observed data. Figure 1 is constructed in the following way.

Plotting σ_D/σ for the two limiting cases indicated in Eqs. 3.13 and 3.15 as a function of D/L* or D/L_F^* in log-log scale, one can obtain a diagram (heavy solid lines) as shown in Fig. 1: From Eq. 3.13b,

$$\log \frac{\sigma_D}{\sigma_{ff}} = \frac{1}{2} \log(1) \qquad D \leq L^* \qquad (4.1)$$

and from Eq. 3.13c

$$\log \frac{\sigma_D}{\sigma_{ff}} = -\frac{1}{2} \log \frac{D}{L^*} \qquad D > L^* \qquad (4.2)$$

Also, from the relationship between σ_D, σ_{F_D} and D which is given by Eq. 2.21,

$$\log \frac{\sigma_D}{\sigma_{ff}} = -\log \frac{D}{L^*} + \log \frac{\sigma_{F_D}}{\sigma_{ff} L^*} \qquad (4.3)$$

where $\sigma_{F_D}^2$ is the variance of the difference process $F_D(x)$ which can be easily calcualted from the observations of F(x).

In Case II, from Eq. 3.15b

$$\log \frac{\sigma_D}{\sigma_{ff}} = \frac{1}{2} \log (1) \qquad D \leq L_F^* \qquad (4.4)$$

and from Eq. 3.15c,

$$\log \frac{\sigma_D}{\sigma_{ff}} = -\log \frac{D}{L_F^*} \qquad D > L_F^* \qquad (4.5)$$

Also from Eqs. 3.6 and 3.27,

$$\log \frac{\sigma_D}{\sigma_{ff}} = \log \frac{\sigma_D \cdot L_F^*}{\sigma_{FF}\sqrt{2}} = -\log \frac{D}{L_F^*} + \log \frac{\sigma_{F_D}}{\sigma_{FF}\sqrt{2}} \qquad (4.6)$$

Due to Eqs. 3.18 and 3.21, along a straight line making a 45° angle with the log D/L^* or log D/L_F^* axis such as A in Fig. 1, the value of σ_{F_D} is constant and hence, the σ_{F_D} axis B can be constructed.

A set of eight samples of the exact σ_D-D relationships (Eqs. 3.13a and 3.15a) are also plotted in Fig. 1 (dashed curves) using the particular forms of the correlation function for the original process f(x) which are designated as Types 2 and 3, respectively in Tables 1 and 2 for Case II, and as Types 1, 2, 5 and 6 in Table 3 for Case I. It can be observed from Fig. 1 that all these curves asymptotically approach the solid lines in the ranges where $D \to 0$ and $D \to \infty$, and that in the intermediate range of D, the solid lines tend to represent the average or upper-bound trend of all the dashed curves. Using a diagram as shown in Fig. 1, the correlation scale L^* or L_F^* can be determined from the length D at the intersection of solid lines in Fig. 1 if such a diagram is constructed as a function of D using the variances estimated from observed data (see Section 5).

It should be noted here that all the pairs of correlation functions and power spectral density functions indicated in Tables 1, 2 and 3 except Types 1 and 2 in Table 3 have at least first-order derivative processes. Hence they are used as the correlation function or power spectral density function of the homogeneous process not only F(x) but also f(x). However, Types 1 and 2 in Table 3 are used only for f(x). Indeed, for Types 1 and 2 in Table 3, only one correlation scale L^* of f(x) ca be defined as shown in Table 4. However, for each of the other correlation functions, we can define the two correlation scales L^* and L_F^* depending on the interpretation of the process (f(x) or F(x)). The correlation scale L_F^* in parentheses in Table 4 are obtained by interpreting the correlation functions in Table 4 as those of the integral process F(x).

For the correlation functions of Types 2-6 in Tables 1 and 2 (Case II), the definition given by Eq. 3.14 leads to a zero correlation scale ($L^* = 0$) as indicated in Table 4. Also, for correlation functions of Types 1 and 2 in Table 3 (Case I), the definition given by Eq. 3.5 leads to a zero correlation scale ($L_F^* = 0$). As shown in this example (Table 4), we must classify the original process f(x) into two cases (Case I and Case II) to obtain a physically meaningful correlation scale for the stochastic process f(x). In this sense, a real phenomenon may be modeled by the Case I process, the Case II process or the combined process of the Case I process and Case II process. By appropriately combining the Case I process and Case II process, we can construct a more sophisticated homogeneous stochastic process model which may be able to more accurately represent real phenomena. However, in this study, we deal with only the fundamental characteristics of the Case I process and the Case II process because they are fundamentals of homogeneous stochastic processes, and for a more sophisticated model, a deep understanding of not only measured data but also both the physical mechanism indicated by the data and the accuracy required for modeling is essential, and hence, this issue is beyond the scope of this study.

5. PRACTICAL ESTIMATION PROCEDURE AND NUMERICAL EXAMPLES

In this section, we describe a practical procedure for estimating the correlation scale (s) from finite-length observed data with numerical examples. Next we exaine briefly the statistical properties of the estimator of the correlation scale based on this procedure.

5.1 Practical Estimation Procedure

The procedure is as follows (see Fig. 1):

(1) From a set of observed data, estimate the mean value $\tilde{m}$ and the variance $\tilde{\sigma}^2_{ff}$;

(2) Obtain a set of averaging process data $f_D(x)$ for several large values of D and calculate the variances $\tilde{\sigma}^2_D$ from them;

(3) Plot a set of averaging deviation ratios $\tilde{\sigma}_D/\tilde{\sigma}_{ff}$ on a log-log scaled graph as a function of D;

(4) If the estimated deviation ratio $\tilde{\sigma}_D/\tilde{\sigma}_{ff}$ follows straight line I, determine the correlation scale L* from the length D of the intersection between horizontal line H and straight line I. If the ratio $\tilde{\sigma}_D/\tilde{\sigma}_{ff}$ follows straight line II, determine the correlation scale L^*_F the length D of the intersection between horizontal line H and straight line II.

5.2 Numerical Examples

The numerical examples in this study are based on digitally simulated stochastic data using the followig equation (Shinozuka and Yang, 1972):

$$f(x) = \sqrt{2} \sum_{n=1}^{N} \sqrt{S_{ff}(\kappa_n)\Delta\kappa} \cos(\kappa_n x + \phi_n) \tag{5.1}$$

where ϕ_n is the stochastic phase angle uniformly distributed between 0 and 2π, $\Delta\kappa = \kappa_u/N$, $\kappa_n = n\Delta\kappa$ and κ_u = upper cut-off wave number.

For numerical examples, the following data are used: Type 1 in Table 2 for $S_{ff}(\kappa)$ with $b = 31.636$ (m); $\kappa_u = 1$ (rad/m) and $\sigma_{ff}^2 = 1$ (m). By using these data, a sample function of f(x) is shown for the distance of 2,000 (m) in Fig. 2. For D = 200, 300, 400, 500 (m), the variances σ_D^2 are calculated and the resulting deviation ratios σ_D/σ_{ff} are plotted in Fig. 3. From Fig. 3, L* is estimated to be about 56 (m). In fact, in this numerical example, the true $L^* = \sqrt{\pi}\ b = 56.07$ (m).

5.3 Statistical Assessments of Correlation Scale Estimates

To establish the quality of the estimator, we will use two principal factors in this study: "Unbiased" and "consistent," that is,

$$E[\tilde{\phi}] = \phi \qquad \text{unbiased} \tag{5.2}$$

and

$$\lim E[(\tilde{\phi} - \phi)^2] = 0 \qquad \text{consistent} \tag{5.3}$$

where $\tilde{\phi}$ is an estimator for the parameter ϕ. The analysis procedures that follow are basically based on those by J.S. Bendat and A.G. Piersol (1971).

5.3.1 Mean Values

Consider the sample record $\tilde{f}(x)$ from a homogeneous (ergodic) stochastic process over a finite length D_0. The mean value can be estimated by

$$\tilde{m} = \frac{1}{D_0} \int_0^{D_0} \tilde{f}(x)dx \tag{5.4}$$

The true mean value is

$$m = E[\tilde{f}(x)] \tag{5.5}$$

Then, the expected value of $\tilde{m}$ is

$$E[\tilde{m}] = \frac{1}{D_0}\int_0^{D_0} E[\tilde{f}(x)]dx = m \tag{5.6}$$

Hence, $\tilde{m}$ is an unbiased estimate of m in accordance with Eq. 5.2.

The variance of $\tilde{m}$ can be expressed as

$$Var[\tilde{m}] = E[(\tilde{m} - m)^2] \tag{5.7}$$

Introducing the covariance function $C_{\tilde{f}\tilde{f}}(\xi)$ of $\tilde{f}(x)$ as

$$\begin{aligned} C_{\tilde{f}\tilde{f}}(\xi) &= E[\{\tilde{f}(x+\xi) - m\}\{\tilde{f}(x) - m\}] \\ &= R_{\tilde{f}\tilde{f}}(\xi) - m^2 \end{aligned} \tag{5.8}$$

where $R_{\tilde{f}\tilde{f}}(\xi)$ is the correlation function of $\tilde{f}(x)$

$$R_{\tilde{f}\tilde{f}}(\xi) = E[\tilde{f}(x+\xi)\tilde{f}(x)] \tag{5.9}$$

then, the variance of $\tilde{m}$ is written in terms of the covariance function as follows:

$$\begin{aligned} Var[\tilde{m}] &= \frac{1}{D_0^2}\int_0^{D_0}\int_0^{D_0} \{E[\tilde{f}(\xi_1)\tilde{f}(\xi_2)] - m^2\}d\xi_1 d\xi_2 \\ &= \frac{1}{D_0^2}\int_0^{D_0}\int_0^{D_0} C_{\tilde{f}\tilde{f}}(\xi_1-\xi_2)d\xi_1 d\xi_2 \\ &= \frac{1}{D_0}\int_{-D}^{D_0} (1 - \frac{|\xi|}{D_0})C_{\tilde{f}\tilde{f}}(\xi)d\xi \end{aligned} \tag{5.10}$$

Similar to Eq. 3.8, Eq. 5.10 is given for $D_0 \to \infty$ by

$$Var[\tilde{m}] = \frac{1}{D_0}\int_{-\infty}^{\infty} C_{\tilde{f}\tilde{f}}(\xi)d\xi \tag{5.11}$$

Hence, $Var[\tilde{m}]$ approaches zero as $D_0 \to \infty$, indicating that $\tilde{m}$ is a consistent estimate of the mean value m.

5.3.2 Variances

Since the mean value m can be estimated unbiasedly and consistently by

Eq. 5.4, in this section we consider the zero mean process f(x) with variance σ_{ff}^2. The variance of f(x)may be estimated by

$$\tilde{\sigma}_{ff}^2 = \frac{1}{D_0} \int_0^{D_0} f^2(x)dx \tag{5.12}$$

The expected value of the estimate $\tilde{\sigma}^2$ is

$$E[\tilde{\sigma}^2] = \frac{1}{D_0} \int_0^{D_0} E[f^2(x)]dx = \sigma_{ff}^2 \tag{5.13}$$

Hence $\tilde{\sigma}_{ff}^2$ is an unbiased estimate of σ_{ff}^2 in accordance with Eq. 5.2.

The variance of the estimate is given by

$$\begin{aligned} Var[\tilde{\sigma}^2] &= E\left[\{\tilde{\sigma}^2 - \sigma^2\}^2\right] \\ &= \frac{1}{D_0^2} \int_0^{D_0}\int_0^{D_0} \{E\left[f^2(\xi_1)f^2(\xi_2)\right] - \sigma_{ff}^4\}d\xi_1 d\xi_2 \end{aligned} \tag{5.14}$$

Assume now that f(x) is a Gaussian stochastic process. Then the expected value in Eq. 5.14 can be expressed in terms of second-order statistics such as

$$E\left[f^2(\xi_1)f^2(\xi_2)\right] = 2R_{ff}^2(\xi_1-\xi_2) + \sigma_{ff}^4 \tag{5.15}$$

Substitution of Eq. 5.15 into Eq. 5.14 yields

$$\begin{aligned} Var[\tilde{\sigma}^2] &= \frac{2}{D_0^2} \int_0^{D_0}\int_0^{D_0} [R_{ff}^2(\xi_1-\xi_2)d\xi_1 d\xi_2 \\ &= \frac{2}{D_0} \int_{-D_0}^{D_0} (1 - \frac{|\xi|}{D_0})R_{ff}^2(\xi)d\xi \end{aligned} \tag{5.16}$$

For large D_0, where $|\xi| << D_0$, the variance becomes

$$Var[\tilde{\sigma}_{ff}^2] = \frac{2}{D_0} \int_{-\infty}^{\infty} R_{ff}^2(\xi)d\xi \tag{5.17}$$

Thus the $\tilde{\sigma}_{ff}^2$ estimated by Eq. 5.12 is a consistent estimate of σ_{ff}^2 because $Var[\tilde{\sigma}_{ff}^2]$ approaches zero as $D_0 \to \infty$ assuming a finite value of the integral.

5.3.3 Estimation of Correlation Scales

For the reason that the mean value and variance estimators $\tilde{m}$ and $\tilde{\sigma}^2$ of $\tilde{f}(x)$ are unbiased and consistent estimates of $f(x)$ as shown in prior sections, we restrict our attention here to processes with zero mean and unit variance. Then the correlation scales L^* and L_F^* may be estimated from Eqs. 3.13c and 3.15c as

$$\tilde{L}^* = D\,\tilde{\sigma}_D^2 \tag{5.18}$$

and

$$\tilde{L}_F^* = D\,\tilde{\sigma}_D \tag{5.19}$$

where $\tilde{\sigma}_D^2$ is an variance estimate of the averaging process $f_D(x)$ of $f(x)$. It may be estimated by

$$\begin{aligned}\tilde{\sigma}_D^2 &= \frac{1}{D_0 - D}\int_{\frac{D}{2}}^{D_0 - \frac{D}{2}} f_D^2(x)dx \\ &= \frac{1}{D_0}\int_0^{D_0} f_D^2(x)dx\end{aligned} \tag{5.20}$$

In Eqs. 5.18-5.20, L^* or $L_F^* << D << D_0$ is assumed.

The expected value of $\tilde{\sigma}_D^2$ is

$$E[\tilde{\sigma}_D^2] = \frac{1}{D_0}\int_0^{D_0} E\left[f_D^2(x)\right]dx = \sigma_D^2 \tag{5.21}$$

where σ_D^2 is the true variance of $f_D(x)$. Then, the expectations of $\tilde{L}^*$ and $\tilde{L}_F^*$ are given by

$$E[\tilde{L}^*] = D\,E[\tilde{\sigma}_D^2] = D\sigma_D^2 = L^* \tag{5.22a}$$

and

$$E[\tilde{L}_F^*] = D\,E[\tilde{\sigma}_D] = D\sigma_D = L_F^* \tag{5.22b}$$

Hence $\tilde{L}^*$ and $\tilde{L}_F^*$ are the unbiased estimates of L^* and L_F^*, respectively.

The variance of $\tilde{\sigma}_D^2$ is given by

$$\mathrm{Var}[\tilde{\sigma}_D^2] = E\left[\{\tilde{\sigma}_D^2 - \sigma_D^2\}^2\right] \tag{5.23}$$

Then, the variances of $\tilde{L}^*$ and $\tilde{L}_F^*$ are expressed such that

$$\mathrm{Var}[\tilde{L}^*] = D^2\mathrm{Var}[\tilde{\sigma}_D^2] \tag{5.24a}$$

and

$$\mathrm{Var}[\tilde{L}_F^*] = D^2\sqrt{\mathrm{Var}[\tilde{\sigma}_D^2]} \tag{5.24b}$$

Similar to Eqs. 5.14-5.17, Eq. 5.23 can be written for $D_0 \to \infty$

$$\mathrm{Var}[\tilde{\sigma}_D^2] = \frac{2}{D_0}\int_{-\infty}^{\infty} R_{f_D}^2(\xi)d\xi \tag{5.25}$$

where $R_{f_D}(\xi)$ is the correlation function of $f_D(x)$ given by

$$\begin{aligned}
R_{f_D}(\xi) &= E\left[f_D(x+\xi)f_D(x)\right] \\
&= \frac{1}{D^2}\int_{x+\xi-\frac{D}{2}}^{x+\xi+\frac{D}{2}} E[f(y)f(z)]dydz \\
&= \frac{1}{D^2}\int_0^D\int_0^D R_{ff}(\xi+\xi_1-\xi_2)d\xi_1 d\xi_2 \\
&= \frac{1}{D}\int_{-D}^{D}\left(1 - \frac{|\xi_0|}{D}\right)R_{ff}(\xi+\xi_0)d\xi_0
\end{aligned} \tag{5.26}$$

If the indefinite integral process $F(x)$ of $f(x)$ is homogeneous, $R_{f_D}(\xi)$ is given by

$$\begin{aligned}
R_{f_D}(\xi) &= \frac{1}{D^2}E\left[\left\{F\left(x+\xi+\frac{D}{2}\right) - F\left(x+\xi-\frac{D}{2}\right)\right\}\left\{F\left(x+\frac{D}{2}\right) - F\left(x-\frac{D}{2}\right)\right\}\right] \\
&= \frac{1}{D^2}\left[2R_{FF}(\xi) - R_{FF}(\xi+D) - R_{FF}(\xi-D)\right]
\end{aligned} \tag{5.27}$$

where $R_{FF}(\xi)$ is the correlation function of $F(x)$. For $D \gg L^*$ or L_F^*, Eqs. 5.26 and 5.27 may be written as

$$R_{f_D}(\xi) = \frac{1}{D} \int_{-\infty}^{\infty} R_{ff}(\xi+\xi_0)d\xi_0 \tag{5.28}$$

and

$$R_{f_D}(\xi) = \frac{2}{D^2} R_{FF}(\xi) \tag{5.29}$$

From Eqs. 5.25, 5.28 and 5.29, the variance of $\tilde{\sigma}_D^2$ is given by

$$\mathrm{Var}[\tilde{\sigma}_D^2] = \frac{2}{D_0 D^2}\, d\xi \Big[\int_{-\infty}^{\infty} R_{ff}(\xi+\xi_0)d\xi_0\Big]^2 \qquad \text{when } S_{ff}(0) \neq 0 \tag{5.30a}$$

and

$$\mathrm{Var}[\tilde{\sigma}_D^2] = \frac{8}{D_0 D^4} \int_{-\infty}^{\infty} R_{FF}^2(\xi)d\xi \qquad \text{when } S_{ff}(0) = 0 \tag{5.30b}$$

Hence the variances of the correlation scale estimates are from Eqs. 5.24 and 5.30,

$$\mathrm{Var}[\tilde{L}^*] = \frac{2}{D_0} \int_{-\infty}^{\infty} R_{ff}(\xi+\xi_0)d\xi_0\Big]^2 \qquad \text{when } S_{ff}(0) \neq 0 \tag{5.31a}$$

and

$$\mathrm{Var}[\tilde{L}_F^*] = \sqrt{\frac{8}{D_0} \int_{-\infty}^{\infty} R_{FF}^2(\xi)d\xi} \qquad \text{when } S_{ff}(0) = 0 \tag{5.31b}$$

Thus $\tilde{L}^*$ and $\tilde{L}_F^*$ given by Eqs. 5.18-5.20 are the consistent estimates of L^* and L_F^* since $\mathrm{Var}[\tilde{L}^*]$ and $\mathrm{Var}[\tilde{L}_F^*]$ approach zero as $D_0 \to \infty$. Equation 5.31a is identical with that derived by E. Vanmarcke (1983). It should again be noted that Eq. 5.31 is derived from the assumption that f(x) is a homogeneous Gaussian process with zero mean and unit variance. Hence, the correlation function

$R_{ff}(\xi)$ in Eq. 5.31a is normalized with $R_{ff}(0) = 1$.

6. CORRELATION SCALES OF TWO-DIMENSIONAL STOCHASTIC FIELDS

6.1 Variance of Averaging Process

In this section, we briefly discuss the correlation scales of two-dimensional stochastic fields by extending the procedures developed in previous chapters. For the original homogeneous stochastic field f(x,y) with zero mean and variance σ_{ff}^2, the averaging field $F_A(x,y)$ may be defined as

$$f_A(x,y) = \frac{1}{A} \int_{x-\frac{D_x}{2}}^{x+\frac{D_x}{2}} \int_{y-\frac{D_x}{2}}^{y+\frac{D_y}{2}} f(u,v)dudv \tag{6.1}$$

where $A = D_x D_y$, and D_x, D_y are the averaging distances of the x- and y-coordinates, respectively. Introducing the following indefinite integrals

$$F_x(x,y) = \int f(x,y)dx \qquad \text{or} \qquad \frac{\partial F_x(x,y)}{\partial x} = f(x,y) \tag{6.2a}$$

$$F_y(x,y) = \int f(x,y)dy \qquad \text{or} \qquad \frac{\partial F_y(x,y)}{\partial y} = f(x,y) \tag{6.2b}$$

and

$$F(x,y) = \iint f(x,y)dxdy \qquad \text{or} \qquad \frac{\partial^2 f(x,y)}{\partial x \partial y} = f(x,y) \tag{6.2c}$$

If they are homogeneous, the power spectral density functions of $F_x(x,y)$, $F_y(x,y)$ and F(x,y) are given by

$$S_{F_xF_x}(\kappa_x,\kappa_y) = \frac{S_{ff}(\kappa_x,\kappa_y)}{\kappa_x^2} \tag{6.3a}$$

$$S_{F_yF_y}(\kappa_x,\kappa_y) = \frac{S_{ff}(\kappa_x,\kappa_y)}{\kappa_y^2} \tag{6.3b}$$

and

$$S_{FF}(\kappa_x,\kappa_y) = \frac{S_{ff}(\kappa_x,\kappa_y)}{\kappa_x^2\kappa_y^2} \tag{6.3c}$$

where $S_{ff}(\kappa_x,\kappa_y)$ is the power spectral density function of f(x,y). As is well known, $S_{ff}(\kappa_x,\kappa_y)$ is related to the correlation function $R_{ff}(\xi_x,\xi_y)$ through the Wiener-Khintchine transform pair:

$$S_{ff}(\kappa_x,\kappa_y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{ff}(\xi_x,\xi_y)e^{-i(\kappa_x\xi_x+\kappa_y\xi_y)}d\xi_x d\xi_y \tag{6.4a}$$

$$R_{ff}(\xi_x,\xi_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{ff}(\kappa_x,\kappa_y)e^{i(\kappa_x\xi_x+\kappa_y\xi_y)}d\kappa_x d\kappa_y \tag{6.4b}$$

where κ_x, κ_y are the wave numbers of the x- and y-coordinates, respectively, and ξ_x, ξ_y are the separation distances of the x- and y-coordinates, respectively. By taking into account the relations $R_{ff} = (\xi_x,\xi_y) = R_{ff}(-\xi_x,\xi_y)$ and $R_{ff}(\xi_x,-\xi_y) = R_{ff}(-\xi_x,\xi_y)$, Eq. 6.4 is also expressed as

$$S_{ff}(\kappa_x,\kappa_y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{ff}(\xi_x,\xi_y)\cos(\kappa_x\xi_x+\kappa_y\xi_y)d\xi_x d\xi_y \tag{6.5}$$

$$R_{ff}(\kappa_x,\kappa_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{ff}(\kappa_x,\kappa_y)\cos(\kappa_x\xi_x+\kappa_y\xi_y)d\kappa_x d\kappa_y \tag{6.6}$$

Similar to the discussions in Section 2, the integral processes $F_x(x,y)$, $F_y(x,y)$ and $F(x,y)$ are not always homogeneous. The conditions of homogeneity of the integral processes depend on the behavior of the power spectral density function $S_{ff}(\kappa_x,\kappa_y)$ of the original process f(x,y) at the origin $\kappa_x = \kappa_y = 0$. Using asymptotic expansion of $\cos(\kappa_x\xi_x+\kappa_y\xi_y)$, $S_{ff}(\kappa_x,\kappa_y)$ can be expressed as

$$S_{ff}(\kappa_x,\kappa_y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{ff}(\xi_x,\xi_y)[1 - \frac{(\kappa_x\xi_x+\kappa_y\xi_y)^2}{2!} + \frac{(\kappa_x\xi_x+\kappa_y\xi_y)^4}{4!} + \cdots]d\xi_x d\xi_y$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{ff}{*}\xi_x,\xi_y)[1 - \frac{1}{2!}(\kappa_x^2\xi_x^2+\kappa_y^2\xi_y^2+2\kappa_x\kappa_y\xi_x\xi_y)$$

$$+ \frac{1}{4!}(\kappa_x^4\xi_x^4+\kappa_y^4\xi_y^4+6\kappa_x^2\kappa_y^2\xi_x^2\xi_y^2 + 4\kappa_x^3\kappa_y\xi_x^3\xi_y+2\kappa_x\kappa_y^3\xi_x\xi_y^3)+\cdots]d\xi_x d\xi_y \tag{6.7}$$

If the process is quadrant symmetric, i.e., $R_{ff}(\xi_x,\xi_y) = R_{ff}(\xi_x,-\xi_y)$, Eq. 5.7

is expressed as

$$S_{ff}(\kappa_x,\kappa_y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{ff}(\xi_x,\xi_y)\Big[1 - \frac{1}{2!}\left(\kappa_x^2\xi_x^2+\kappa_y^2\xi_y^3\right)$$
$$+ \frac{1}{4!}\left(\kappa_x^4\xi_x^4+\kappa_y^4\xi_y^4+6\kappa_x^2\kappa_y^2\xi_x^2\xi_y^2\right)\cdots\Big]d\xi_x d\xi_y \tag{6.8}$$

To simplify the analysis that follows, we will consider the quadrant symmetric process.

From Eqs. 6.3 and 6.8, the conditions of homogeneity of the integral processes $F_x(x,y)$, $F_y(x,y)$ and $F(x,y)$ may be summarized as follows:

Case 1:	$S_{ff}(0,0) \neq 0$	$F_x(x,y)$, $F_y(x,y)$ and $F(x,y)$ are all nonhomogeneous	
Case 2:	$S_{ff}(0,0) =$	$S_{ff}^{xx}(0,0) = S_{ff}^{yy}(00) = 0$	$F(x,y)$ is homogeneous
Case 3:	$S_{ff}(0,0) = S_{ff}^{yy}(0,0) = 0$ and $S_{ff}^{xx}(0,0) \neq 0$		$F_x(x,y)$ is homogeneous
Case 4:	$S_{ff}(0,0) = S_{ff}^{xx}(0,0) = 0$ and $S_{ff}^{yy}(0,0) \neq 0$		$F_y(x,y)$ is homogeneous

where $S_{ff}^{xx}(0,0)$ and $S_{ff}^{yy}(0,0)$ are the second derivative values of $S_{ff}(\kappa_x,\kappa_y)$ at the origin given by

$$S_{ff}^{xx}(0,0) = \frac{\partial^2 S_{ff}(\kappa_x,\kappa_y)}{\partial\kappa_x^2}\bigg|_{\kappa_x=\kappa_y=0} = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\xi_x^2 R_{ff}(\xi_x,\xi_y)d\xi_x d\xi_y \tag{6.9a}$$

$$S_{ff}^{yy}(0,0) = \frac{\partial^2 S_{ff}(\kappa_x,\kappa_y)}{\partial\kappa_y^2}\bigg|_{\kappa_x=\kappa_y=0} = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\xi_y^2 R_{ff}(\xi_x,\xi_y)d\xi_x d\xi_y \tag{6.9b}$$

Similar to Eq. 2.14, the variance σ_A^2 of $f_A(x,y)$ is given such that:

$$\sigma_A^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{\sin\frac{\kappa_x D_x}{2}}{\frac{\kappa_x D_x}{2}}\right]^2\left[\frac{\sin\frac{\kappa_y D_y}{2}}{\frac{\kappa_y D_y}{2}}\right]^2 S_{ff}(\kappa_x,\kappa_y)d\kappa_x d\kappa_y \tag{6.10a}$$

$$= \frac{1}{A}\int_{-D_x}^{D_x}\int_{-D_y}^{D_y}\left(1-\frac{|\xi_x|}{D_x}\right)\left(1-\frac{|\xi_y|}{D_y}\right)R_{ff}(\xi_x,\xi_y)d\xi_x d\xi_y \tag{6.10b}$$

For Case 2 where $f(x,y)$ is homogeneous, Eq. 6.10 becomes

$$\sigma_A^2 = \frac{16}{A^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{S_{ff}(\kappa_x,\kappa_y)}{\kappa_x^2\kappa_y^2}(\sin\frac{\kappa_x D_x}{2})^2(\sin\frac{\kappa_y D_y}{2})^2 d\kappa_x d\kappa_y$$

$$= \frac{16}{A^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{FF}(\kappa_x,\kappa_y)\{\frac{1-\cos\kappa_x D_x}{2}\}\{\frac{1-\cos\kappa_y D_y}{2}\} d\kappa_x d\kappa_y$$

$$= \frac{4}{A^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{FF}(\kappa_x,\kappa_y)\{1-\cos\kappa_x D_x-\cos\kappa_y D_y+\cos\kappa_x D_x\cos\kappa_y D_y\} d\kappa_x d\kappa_y$$

$$= \frac{4}{A^2}\left[R_{FF}(0,0)+R_{FF}(D_x,D_y)-R_{FF}(D_x,0)-R_{FF}(0,D_y)\right] \quad (6.11)$$

In the derivation of Eq. 6.11, the following Wiener-Khintchine relationships for a quadrant process are used:

$$S_{ff}(\kappa_x,\kappa_y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{ff}(\xi_x,\xi_y)\cos\kappa_x\xi_x\cos\kappa_y\xi_y d\xi_x d\xi_y$$

$$R_{ff}(\xi_x,\xi_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{ff}(\kappa_x,\kappa_y)\cos\kappa_x\xi_x\cos\kappa_y\xi_y d\kappa_x d\kappa_y \quad (6.12b)$$

For Case 3 where $F_x(x,y)$ is homogeneous, the variance σ_A^2 given by Eq. 6.10 is

$$\sigma_A^2 = (\frac{2}{D_x})^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{S_{ff}(\kappa_x,\kappa_y)}{\kappa_x^2}[\sin\frac{\kappa_x D_x}{2}]^2[\frac{\sin\frac{\kappa_y D_y}{2}}{\frac{\kappa_y D_y}{2}}]^2 d\kappa_x d\kappa_y$$

$$= (\frac{2}{D_x})^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{F_xF_x}(\kappa_x,\kappa_y)[\sin\frac{\kappa_x D_x}{2}]^2[\frac{\sin\frac{\kappa_y D_y}{2}}{\frac{\kappa_y D_y}{2}}]^2 d\kappa_x d\kappa_y$$

$$= \frac{2}{D_x^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{F_yF_y}(\kappa_x,\kappa_y)[1-\cos\kappa_y D_y][\frac{\sin\frac{\kappa_x D_x}{2}}{\frac{\kappa_x D_x}{2}}]^2 d\kappa_x d\kappa_y \quad (6.13)$$

For Case 4 where $F_y(x,y)$ is homogeneous, Eq. 6.10 becomes:

$$\sigma_A^2 = \frac{2}{D_y^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{F_yF_y}(\kappa_x,\kappa_y)[1-\cos\kappa_y D_y][\frac{\sin\frac{\kappa_x D_x}{2}}{\frac{\kappa_x D_x}{2}}]^2 d\kappa_x d\kappa_y \tag{6.14}$$

6.2 Definitions of Correlation Scales of Two-Dimensional Processes

When $D_x = D_y \to 0$, the variance σ_A^2 given by Eq. 6.10 approaches σ_{ff}^2 since the window function in Eq. 6.10 approaches one. That is,

$$\sigma_A^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{ff}(\kappa_x,\kappa_y)d\kappa_x d\kappa_y = \sigma_{ff}^2 \qquad \text{as } D_x = D_y \to 0 \tag{6.15}$$

On the other hand, for $D_x = D_y \to \infty$, the variance σ_A^2 takes the following forms depending on the behavior of the power spectral density function $S_{ff}(\kappa_x,\kappa_y)$ of $f(x,y)$ at the origin.

For Case 1: from Eq. 6.10 when $D_x = D_y \to \infty$,

$$\begin{aligned}
\sigma_A^2 &= \frac{4}{D_xD_y} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\frac{\sin u}{u}]^2[\frac{\sin v}{v}]^2 S_{ff}(\frac{2u}{D_x},\frac{2v}{D_y})dudv \\
&= \frac{4}{D_xD_y} S_{ff}(0,0) \int_{-\infty}^{\infty} [\frac{\sin u}{u}]^2 du \int_{-\infty}^{\infty} [\frac{\sin v}{v}]^2 dv \\
&= \frac{(2\pi)^2}{D_xD_y} S_{ff}(0,0) \\
&= \frac{1}{D_xD_y} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{ff}(\xi_x,\xi_y)d\xi_x d\xi_y
\end{aligned} \tag{6.16}$$

For Case 2: from Eq. 6.11,

$$\sigma_A^2 = \frac{4}{(D_xD_y)^2} R_{FF}(0,0) = \frac{4}{(D_xD_y)^2} \sigma_{FF}^2 \qquad \text{as } D_x = D_y \to \infty \tag{6.17}$$

where σ_{FF}^2 is the variance of $F(x,y)$.

For Case 3: from Eq. 6.13,

$$\begin{aligned}
\sigma_A^2 &= \frac{2}{D_x^2} \cdot \frac{2}{D_y} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{F_xF_x}(\kappa_x,\frac{2v}{D_y})[1-\cos\kappa_x D_x][\frac{\sin v}{v}]^2 d\kappa_x dv \\
&= \frac{2}{D_x^2} \cdot \frac{2}{D_y} \int_{-\infty}^{\infty} S_{F_xF_x}(\kappa_x,0)[1-\cos\kappa_x D_x] \int_{-\infty}^{\infty} [\frac{\sin v}{v}]^2 dv d\kappa_x
\end{aligned}$$

$$= \frac{2}{D_x^2} \cdot \frac{2\pi}{D_y} \int_{-\infty}^{\infty} S_{F_xF_x}(\kappa_x,0)[1-\cos\kappa_xD_x]d\kappa_x$$

$$= \frac{2}{D_x^2} \cdot \frac{2\pi}{D_y} R_{F_xF_x}(0) \qquad \text{as } D_x = D_y \to \infty \tag{6.18}$$

where $R_{F_xF_x}(\xi_x)$ is defined such that

$$R_{F_xF_x}(\xi_x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{F_xF_x}(\xi_x,\xi_y)d\xi_y \tag{6.19}$$

then

$$S_{F_xF_x}(\kappa_x,0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{F_xF_x}(\xi_x)\cos \kappa_x\xi_x d\xi_x \tag{6.20}$$

the inverse transform reclaims

$$R_{F_xF_x}(\xi_x) = \int_{-\infty}^{\infty} S_{F_xF_x}(\kappa_x,0)\cos \kappa_x\xi_x d\kappa_x \tag{6.21}$$

For Case 4: from Eq. 6.14,

$$\sigma_A^2 = \frac{2\pi}{D_x} \cdot \frac{2}{D_y^2} R_{F_yF_y}(0) \qquad \text{as } D_x = D_y \to \infty \tag{6.22}$$

where $R_{F_yF_y}(\xi_y)$ is defined by

$$\begin{aligned} R_{F_yF_y}(\xi_y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{F_yF_y}(\xi_x,\xi_y)d\xi_x \\ &= \int_{-\infty}^{\infty} S_{F_yF_y}(0,\kappa_y)\cos \kappa_y\xi_y d\kappa_y \end{aligned} \tag{6.23}$$

and

$$S_{F_yF_y}(0,\kappa_y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{F_yF_y}(\xi_y)\cos \kappa_y\xi_y d\xi_y \tag{6.24}$$

Summarizing Eqs. 6.15-6.18 and 6.22, the correlation scales of f(x,y) can be defined as follows.

For Case 1:

$$\sigma_A^2 = \sigma^2 \qquad \text{as } D_x = D_y \to 0 \tag{6.25a}$$

$$= \frac{A^*}{D_x D_y}\, \sigma^2 = \left(\frac{\bar{A}^*}{A}\right)\sigma^2 \qquad \text{as } D_x = D_y \to \infty \qquad (6.25b)$$

where A^* is the correlation scale of $f(x,y)$ for Case 1 defined by

$$A^* = \frac{1}{\sigma^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{ff}(\xi_x, \xi_y)\, d\xi_x d\xi_y = \frac{2\pi}{\sigma^2}\, S_{ff}(0,0) \qquad (6.26)$$

For Case 2:

$$\sigma_A^2 = \sigma^2 \qquad \text{as } D_x = D_y \to 0 \qquad (6.27a)$$

$$= \left(\frac{A_F^*}{D_x D_y}\right)^2 \sigma^2 = \left(\frac{A_F^*}{A}\right)^2 \sigma^2 \qquad \text{as } D_x = D_y \to \infty \qquad (6.27b)$$

where A_F^* is the correlation scale of $f(x,y)$ for Case 2 defined by

$$A_F^* = 2\, \frac{\sigma_{FF}}{\sigma_{ff}} = \frac{1}{2\pi^2}\, A_F \qquad (6.28a)$$

$$A_F = (2\pi)^2\, \frac{\sigma_{FF}}{\sigma_{ff}} = (2\pi)^2 \sqrt{\frac{R_{FF}(0,0)}{R_{FF}''''(0,0)}} = (2\pi)^2 \sqrt{\frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{FF}(\kappa_x, \kappa_y)\, d\kappa_x d\kappa_y}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \kappa_x^2 \kappa_y^2 S_{FF}(\kappa_x, \kappa_y)\, d\kappa_x d\kappa_y}} \qquad (6.28b)$$

For Case 3:

$$\sigma_A^2 = \sigma^2 \qquad \text{as } D_x = D_y \to 0 \qquad (6.29a)$$

$$= \frac{L_x^{*3}}{D_x^2 D_y}\, \sigma^2 \qquad \text{as } D_x = D_y \to \infty \qquad (6.29b)$$

where L_x^* is the correlation distance of $f(x,y)$ for Case 3 defined by

$$L_x^* = \left[\frac{2}{\sigma_{ff}^2} \int_{-\infty}^{\infty} R_{F_x F_x}(0, \xi_y)\, d\xi_y\right]^{1/3} \qquad (6.30a)$$

$$= \left[4\pi\, \frac{\int_{-\infty}^{\infty} S_{F_x F_x}(\kappa_x, 0)\, d\kappa_x}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \kappa_x^2 S_{F_x F_x}(\kappa_x, \kappa_y)\, d\kappa_x d\kappa_y}\right]^{1/3} \qquad (6.30b)$$

The equivalent area of f(x,y) for Case 3 may be defined by

$$A_x^* = L_x^{*2} \tag{6.30c}$$

For Case 4:

$$\sigma_A^2 = \sigma^2 \qquad \text{as } D_x = D_y \to 0 \tag{6.31a}$$

$$= \frac{L_y^{*3}}{D_x D_y}\sigma^2 \qquad \text{as } D_x = D_y \to \infty \tag{6.31b}$$

where L_y^* is the correlation scale of f(x,y) for Case 4 given by

$$L_y^* = [\frac{2}{\sigma^2}\int_{-\infty}^{\infty} R_{F_yF_y}(\xi_x,0)d\xi_x]^{1/3} \tag{6.32a}$$

$$= [4\pi \frac{\int_{-\infty}^{\infty} S_{F_yF_y}(0,\kappa_y)d\kappa_y}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \kappa_y^2 S_{F_yF_y}(\kappa_x,\kappa_y)d\kappa_x d\kappa_y}]^{1/3} \tag{6.32b}$$

The equivalent correlation area may also be defined by

$$A_y^* = L_y^{*2} \tag{6.32c}$$

If we consider the special case where $D_x = D_y = D$, the above results become so simple that the following results may be useful for th estimation of the correlation scales (areas) of f(x,y) using the graphical mehtod indicated in Chapter 4 for one-dimensional stochastic processes.

For Case 1:

$$\sigma_A^2 = \sigma^2 \qquad \text{as } D \to 0 \tag{6.33a}$$

$$= (\frac{L^*}{D})^2\sigma^2 \qquad \text{as } D \to \infty \tag{6.33b}$$

where L* is the equivalent correlation distance of the correlation scale A* of f(x,y) defined by

$$L^* = \sqrt{A^*} \tag{6.33c}$$

For Case 2:

$$\sigma_A^2 = \sigma^2 \qquad \text{as } D \to 0 \tag{6.34a}$$

$$= \left(\frac{L_F^*}{D}\right)^4 \sigma^2 \qquad \text{as } D \to \infty \tag{6.34b}$$

where L_F^* is also the equivalent correlation distance of the correlation scale A_F^* of f(x,y) defined by

$$L_F^* = \sqrt{A_F^*} \tag{6.34}$$

For Case 3:

$$\sigma_A^2 = \sigma^2 \qquad \text{as } D \to 0 \tag{6.35a}$$

$$= \left(\frac{L_x^*}{D}\right)^3 \sigma^2 \qquad \text{as } D \to \infty \tag{6.35b}$$

The equivalent correlation area A_x^* of f(x,y) for Case 3 is given by Eq. 6.30c.

For Case 4:

$$\sigma_A^2 = \sigma^2 \qquad \text{as } D \to 0 \tag{6.36a}$$

$$= \left(\frac{L_y^*}{D}\right)^3 \sigma^2 \qquad \text{as } D \to \infty \tag{6.36b}$$

The equivalent correlation area A_y^* is also given by Eq. 6.32c.

Although Eqs. 6.35b and 6.36b have the same form, it is easy to distinguish them. In fact, if we select a rectangular area where $D_x = 4D_y = D$ for example, from Eqs. 6.29b and 6.31b, a difference appears between Eqs. 6.35b and 6.36b such that $4(L_x^*/D)^3\sigma^2$ for Case 3 and $16(L_y^*/D)^3\sigma^2$ for Case 4. By this difference, we can distinguish between Cases 3 and 4. In Fig. 1, the approximate relationships between σ_A and D given by Eqs. 6.33-6.36 are shown by solid lines. In the same figure, three examples of exact σ_A-D relationships (Eqs. 6.10a, 6.11, 6.13 or 6.14) are plotted (dashed curves) using the following particular forms (separable types) of the correlation function or power spectral density function which satisfy quadrant symmetric conditions.

<u>Power spectrum for Case 1:</u>

$$S_{ff}(\kappa_x,\kappa_y) = \frac{\sigma_{ff}^2}{4\pi} b_x b_y \exp[-(\frac{b_x\kappa_x}{2})^2 - (\frac{b_y\kappa_y}{2})^2] \tag{6.37a}$$

$$A^* = \pi b_x b_y, \qquad L^* = \sqrt{A^*} \tag{6.37b}$$

<u>Power spectrum for Case 2:</u>

$$S_{ff}(\kappa_x,\kappa_y) = \frac{\sigma_{ff}^2}{16\pi} b_x^3 b_y^3 \kappa_x^2 \kappa_y^2 \exp[-(\frac{b_x\kappa_x}{2})^2 - (\frac{b_y\kappa_y}{2})^2] \tag{6.38a}$$

$$A_F^* = b_x b_y, \qquad L_F^* = \sqrt{A_F^*} \tag{6.38b}$$

<u>Power spectrum for Case 3:</u>

$$S_{ff}(\kappa_x,\kappa_y) = \frac{\sigma_{ff}^2}{8\pi} b_x^3 b_y \kappa_x^2 \exp[-(\frac{b_x\kappa_x}{2})^2 - (\frac{b_y\kappa_y}{2})^2] \tag{6.39a}$$

$$L_x^* = [\sqrt{\pi}\, b_x^2 b_y]^{1/3}, \qquad A_x^* = L_x^{*2} \tag{6.39b}$$

<u>Power spectrum for Case 4:</u>

$$S_{ff}(\kappa_x,\kappa_y) = \frac{\sigma_{ff}^2}{8\pi} b_x b_y^3 \kappa_y^2 \exp[-(\frac{b_x\kappa_x}{2})^2 - (\frac{b_y\kappa_y}{2})^2] \tag{6.40a}$$

$$L_y^* = [\sqrt{\pi}\, b_x b_y^2]^{1/3}, \qquad A_y^* = L_y^{*2} \tag{6.40b}$$

It can be observed from Fig. 1 that all these curves asymptotically approach the solid lines in the ranges where $D \to 0$ and $D \to \infty$, and that in the intermediate range of D, the solid lines tend to represent the upper bound of all the dashed curves.

6.3 Numerical Examples

In order to visually illustrate the correlation scales and patterns of variation of $f(x,y)$, we pr3sent in this section some numerical examples simulated by the following equations for quadrant symmetric processes (Shinozuka and Harada, 1986).

$$f(x,y) = \sqrt{2} \sum_{m=1}^{M} \sum_{n=1}^{N} \sqrt{2S_{ff}(\kappa_{x_m}, \kappa_{y_n})\Delta\kappa_x \Delta\kappa_y}$$

$$\times \left[\cos(\kappa_{x_m} x + \kappa_{y_n} y + \phi_{1mn}) + \cos(\kappa_{x_m} x - \kappa_{y_n} y + \phi_{2mn})\right] \qquad (6.40a)$$

$$\Delta\kappa_x = \frac{\kappa_{xu}}{M}, \; \Delta\kappa_y = \frac{\kappa_{yu}}{N}, \; \kappa_{x_m} = m\Delta\kappa_x, \; \kappa_{y_n} = n\Delta\kappa_y \qquad (6.40b)$$

where ϕ_{1mn} and ϕ_{2mn} are independent random phase angles uniformly distributed between 0 and 2π. κ_{xu} and κ_{yu} are the upper cut-off wave numbers of κ_x and κ_y, respectively.-

Example 1: This example is for Case 1 using the power spectrum given by Eq. 6.37 together with the following data: $\sigma = 1$, $b_x = 1.0$ (m), $b_y = 1/\sqrt{2}$ (m), $M = N = 64$, $\kappa_{xu} = \kappa_{yu} = 2\pi$ (rad/m). A sample function of $f(x,y)$ and the size of the correlation area A^* in this example are shown in Fig. 2. In Case 1, the correlation area A^* defined by Eq. 6.26 may signify that the correlation of $f(x,y)$ is extremely high within the size of this area A^* ($= \pi b_x b_y = 2.22\ m^2$).

Example 2: For Case 2 using the power spectrum in Eq. 6.38 together with the following data: $\sigma = 1$, $b_x = b_y = \pi/5$ (m), $M = N = 64$, $\kappa_{xu} = \kappa_{yu} = 4\pi$ (rad/m). In Fig. 3, the size of the correlation area A_F^* ($= b_x b_y = 0.4\ m^2$) in this example and a sample function are shown. For Case 2, the correlation area A_F^* defined by Eq. 6.28 may also be useful for representing the size of the area within which highly correlated observations are made.

Example 3: This is an exmaple for Case 3 using Eq. 6.39 with the following data: $\sigma = 1.24$ (cm), $b_x = 1.131 \times 10^3$ (m), $b_y = 3.012 \times 10^3$ (m), $M = N = 64$, $\kappa_{xu} = 10/b_x = 8.84 \times 10^{-3}$ (rad/m), $\kappa_{yu} = 10/b_y = 3.32 \times 10^{-3}$ (rad/m). The size of the correlation distance L_x^* ($= [\sqrt{\pi}\ b_x^2 b_y]^{1/3} = 1897.4$ m) in this example and a sample function of $f(x,y)$ are shown in Fig. 4. In this case, relatively rapid variation along the x-axis is observed, compared with variation along the y-axis. To arepresent this variation along the x-axis, the correlation distance L_x^* defined by Eq. 6.30 may be suitable. However, the significance of the equivalent area ($A_x^* = L_x^{*2}$) defined by Eq. 6.30c is quite vague in this example.

In conclusion, the correlation areas A^* and A_F^* defined by Eqs. 6.26 ad 6.28 are useful for measuring the area size within which highly correlated data is observed. The correlation distances L_x^* and L_y^* defined by Eqs. 6.30 and 6.32 may also be used as measures of highly correlated distances along the x- and y-axes, respectively, in two dimensional stochastic variation problems.

7. SOME NEW APPLICATION EXAMPLES OF CORRELATION SCALES

7.1 Peak Mean Factor

As briefly described in Section 1.2, the correlation scale has been successfully used in many engineering fields as a measure of approximately obtaining the equivalent number of independent observations from stochastic process data with finite intervals. In the context of this interpretation of the correlation scale, we present here a new approximate observation of the probabiolity distribution of maximum values of stochastic processes.

In many applications of stochastic process ehtory to the analysis and design of structures, a central question is as follows: What is the absolute maximum falue of f(x) with zero mean over the range $0 \leq x \leq L$ where the correlation function or the power spectral density function is known. If the absolute maximum value Y is expressed as $p\sigma_{ff}$, where σ_{ff} is the standard deviation of f(x), and p is the peak stochastic factor, the mean and standard deviation of p is given by (Davenport, 1964):

$$E[p] = \sqrt{2\ \ell n\left(\frac{2L}{L_f}\right)} + \frac{0.577215}{\sqrt{2\ \ell n\left(\frac{2L}{L_f}\right)}} \tag{7.1}$$

$$\sigma_{pp} = \frac{\pi}{\sqrt{6}} \frac{1}{\sqrt{2\ \ell n\left(\frac{2L}{L_f}\right)}} \tag{7.2}$$

where L_f is the apparent wave length defined by

$$L_f = 2\pi \frac{\sigma_{ff}}{\sigma_{\dot{f}\dot{f}}} = 2\pi\sqrt{-\frac{R_{ff}(0)}{R''_{ff}(0)}} = 2\pi\sqrt{\frac{\int_{-\infty}^{\infty} S_{ff}(\kappa)d\kappa}{\int_{-\infty}^{\infty} \kappa^2 S_{ff}(\kappa)d\kappa}} \tag{7.3}$$

Equations 7.1-7.3 assume the existence of L_f defined by Eq. 7.3. However, when the derivative $f'(x)$ of $f(x)$ does not exist, L_f become zero (Case I process). In this case, the above equations are useless. Hence, we need another stable expression for the peak factor p. The following equations for peak factors are based on the the combination of the largest value distribution function (the first type) and the correlation scales A and C indicated in Table 1-1.

For the probability density function of the local maxima X (local points) of a homogeneous Gaussian process, the general expression is well known as (Cartwright and Languet-Higgins, 1956)

$$F_X(y) = \frac{\varepsilon}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{y^2}{2\varepsilon^2\sigma^2}\right\}$$
$$+ \frac{y\sqrt{1-\varepsilon^2}}{\sigma^2} \exp\left\{-\frac{y^2}{2\sigma^2}\right\} F_X^N\left(\frac{y\sqrt{1-\varepsilon^2}}{\sigma\varepsilon}\right) \quad (7.4)$$

where $F_X^N(\cdot)$ is a normal distribution function and ε is the irregularity factor and lies between 0 and 1. For $\varepsilon = 0$ (completely narrow band process), the first term vanishes and Eq. 7.4 reduces to the Rayleigh distribution such that

$$f_x^R(y) = \frac{y}{\sigma^2} \exp\left(-\frac{y^2}{2\sigma^2}\right) \quad (7.5a)$$

with distribution function $F_x^R(y)$

$$F_x^R(y) = 1 - \exp\left(-\frac{y^2}{2\sigma^2}\right) \quad (7.5b)$$

For $\varepsilon = 1$ (completely wide band process), only the first term remains and Eq. 7.4 becomes a Gaussian distribution with zero-mean and variance σ^2 such that

$$f_x^N(y) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right) \quad (7.6)$$

The distribution function is denoted by $F_x^N(y)$. The relationship between the local peak probability density function $f_X(y)$ and the homogeneous (ergodic) process $f(x)$ is schematically illustrated in Fig. 1.

On the other hand, using the exact distribution function $F_Y(y)$ for the greatest peak values among $(X_1, X_2, \ldots, X_n)$ that are statistically independent and identically distributed with $F_X(y)$ as the initial variate X such that (Gumbel, 1958),

$$\begin{aligned} F_Y(y) &= P_r[Y \leqq y] \\ &= P\left[X_1 \leqq y,\ X_2 \leqq y,\ \ldots,\ X_n \leqq y\right] \\ &= \left[F_X(y)\right]^n \end{aligned} \tag{7.7}$$

we may have an approximate distribution function for the "greatest peaks" of the stochastic process $f(x)$ over the range $0 \leqq x \leqq L$ when in Eq. 7.7 we interpret $F_X(x)$ as the distribution function of the distribution density function $f_x(y)$ given by Eq. 7.4 with n given as follows:

$$n = \frac{L}{L^*} \quad \text{or} \quad \frac{L}{L_f^*} \tag{7.8}$$

where L^* and L_f^* are the correlation scales (A and C in Table 1-1) such that

$$L^* = \frac{1}{\sigma^2} \int_{-\infty}^{\infty} R_{ff}(\xi)d\xi \tag{7.9a}$$

and

$$L_f^* = \frac{1}{\sqrt{2}\ \pi} L_f = \sqrt{-\frac{2R_{ff}(0)}{R''_{ff}(0)}} \tag{7.9b}$$

where L_f is the apparent wave length defined by Eq. 7.3. In Eq. 7.8, n signifies the equivalent number of independent observations contained in the interval L since the correlation scales L^* and L_f^* are the measures of the highly correlated length of $f(x)$. Finally, taking into account the fact that the peaks and troughs generally tend to appear as the same number over a finite length, the approximate distribution function $F_Y^A(y)$ for the absolute maximum value of f(x) over the range $0 \leq x \leq L$ may be given by Eq. 7.7 with the following n instead of the n given by Eq. 7.8:

$$n = \frac{2L}{L^*} \text{ or } \frac{2L}{L_f^*} \tag{7.10}$$

As is well known, for large n and exponential type initial function $F_X(y)$, Eq. 7.7 has the Type I asymptotic form classified by Gumbel (1958) such that

$$F_Y(y) = \exp[- e^{-\alpha_m(y-u_n)}] \tag{7.11a}$$

where u_m = the characteristic largest value of the initial variate X and α_m = an inverse measure of the dispersion of Y which are determined by

$$F_X(u_n) = 1 - \frac{1}{n} \quad \text{and} \quad \alpha_n = nf_X(u_n) \tag{7.11b}$$

The mean value E[Y] and standard deviation σ_Y of Y are also given such that

$$E[Y] = u_n + \frac{0.577215}{\alpha_n} \tag{7.12a}$$

$$\sigma_Y = \frac{\pi}{\sqrt{6}} \frac{1}{\alpha_n} \tag{7.12b}$$

As the two extreme cases where $F_X(y) = F_X^R(y)$ and $F_X(y) = F_X^N(y)$, Eq. 7.12 becomes as follows:

For a Raleigh distribution $F_X^R(y)$:

$$E[P] = \frac{E[Y]}{\sigma} = \sqrt{2 \ln n} + \frac{0.577215}{\sqrt{2 \ln n}} \tag{7.13a}$$

$$\sigma_P = \frac{\sigma_Y}{\sigma} = \frac{\pi}{\sqrt{6}} \frac{1}{\sqrt{2 \ln n}} \tag{7.13b}$$

For a normal distribution $F_X^N(y)$:

$$E[P] = \frac{E[Y]}{\sigma} = \sqrt{2\ \ell n\ n} - \frac{\ell n(\ell n\ n) + \ell n\ 4\pi}{2\sqrt{2\ \ell n\ n}} + \frac{0.577215}{\sqrt{2\ \ell n\ n}} \tag{7.14a}$$

$$\sigma_P = \frac{\sigma_Y}{\sigma} = \frac{\pi}{\sqrt{6}} \frac{1}{\sqrt{2\ \ell n\ n}} \tag{7.14b}$$

The mean peak factors given by Eqs. 7.13a and 7.14a are plotted by solid curves as a function of n in Fig. 2. The dashed curves in Fig. 2 are the results from Eq. 7.7. From Fig. 2, Eqs. 7.13 and 7.14 may be used for $n > 10$. For small values of n, Eqs. 7.13a and 7.14a tend to give a larger value of E[P]. Since for the intermediate value of the irregularity factor ε, the value of E[P]. Since, for the intermediate values of the irregularity factor ε, the values of E[P] may lie between the two extreme cases (two solid curves and dashed curves in Fig. 2) with the Rayleigh distribution and normal distribution as the initial distribution $F_X(y)$, a more simple approximation may be appropriate for the mean value of the peak factor:

$$E[P] = \begin{cases} \sqrt{2\ \ell n\ n} & n \geq 1.65 \\ 1 & \text{otherwise} \end{cases} \tag{7.15}$$

Equation 7.15 is also plotted by a dashed curve in Fig. 2 indicating the approximate behavior of E[P].

It is observed from the above discussion that the mean value and standard deviation of the peak factor P derived by Davenport (Eqs. 7.1 and 7.2) are identical with those of Eq. 7.13 with n_D such that

$$n_D = \frac{2L}{L_f} = \frac{1}{\sqrt{2}\ \pi} \cdot \frac{2L}{L_f^*} = \frac{1}{4.443} \cdot \frac{2L}{L_f^*} \tag{7.16}$$

The scaling factor $1/\sqrt{2}\ \pi$ in Eq. 7.16 is due to the fact that, for a narrow band process, the peaks and troughs tend to appear twice within the apparent wave length L_f which is longer than the correlation distance L_f^* ($L_f = \sqrt{2}\ \pi\ L_f^*$) as shown in Fig. 1. However, the effect of n on E[P] is not so sensitive that the difference is small between E[P] with n given by Eqs. 7.10 and 7.16 as shown in Fig. 2. In fact, for example, for n = 40 in Eq. 7.10, Eq. 7.16 gives n_D = 40/4.443 = 9.0. From Fig. 2, the corresponding mean peak factors are read as E[P] = 2.9 for n = 40 and E[P] = 2.4 for n_D = 9.0 indicating little difference. In turn, for a wide band process where $L_f^* = 0$, the number n = 2L/L* may tend to give smaller values than the true number of peak values. In fact, for pure wide band processes (ε = 1) where the correlation function is expressed by the Dirac delta function, the correlation scale becomes a finite value of $2\pi S_{ff}(0)$. However, within this interval $L^* = 2\pi S_{ff}(0)$, true peaks may tend to occur more than once. Hence, the mean value of the peak factor given by eq. 7.14a with n = 2L/L* may give a lower value of E[P].

In conclusion, for practical use of the peak factor of the absolute maximum value of f(x) over the range $0 \leq x \leq L$, as a conservative mean peak factor, Eq. 7.13a may be appropriate with n given by Eq. 7.10, and Eq. 7.14a with n given by Eq. 7.10 as a lower value of the mean peak factor. For more simplicity, Eq. 7.15 may be useful with the n given by Eq. 7.10.

7.2 Seismic Ground RMS Estimate

In contrast to the earthquake-resistant design of above-ground structures where the inertial forces induced by ground acceleration are the main consideration, the spatial variation of the ground motion is of primary importance for buried lifeline structures such as pipelines and tunnels. Consequently, the ground strains and relative displacements between two points along pipelines play main roles in the seismic design of such buried lifeline structures.

Applying stochastic process theory, we can estimate the rms (root mean square) values σ_{u_D} and σ_ε of the relative displacements between two points on a ground surface and the ground strain along the pipe axis at time instant $t = t_0$ form the following equations:

$$\sigma_{u_D} = \sqrt{2}\,\frac{D}{L_u^*}\,\sigma_{uu} \qquad D \leqq L_u^* \qquad (7.17a)$$

$$= \sqrt{2}\,\sigma_{uu} \qquad D > L_u^* \qquad (7.17b)$$

$$\sigma_\varepsilon = \frac{\sqrt{2}}{L^*}\,\sigma_{uu} \qquad D \leqq L_u^* \qquad (7.18a)$$

$$= \frac{\sqrt{2}}{D}\,\sigma_{uu} \qquad D > L_u^* \qquad (7.18b)$$

where D is the relative distance betwen two points and L_u^* is the correlation distance of the seismic ground displacement $u(x,t_0)$ at $t = t_0$ denoted by C in Table 1-1. In Eqs. 7.17 and 7.18, the parameters are only σ_{uu} of the ground displacement, the correlation distance L_u^* and relative distance D between two points on the ground surface. More details and a field data analysis can be seen in the paper by Harada and Shinozuka (1986).

7.3 Miscellanea

In a digital time (spatial) series analysis and simulation, we must determine the upper cut-off frequency ω_u (upper cut-off wave number κ_u) above which the power spectral density function is considered to be zero. Fo5r this upper cut-off wave number κ_u, the spectral scales defined by Eq. 3.16 may be used as a measure of ω_u (κ_u) such that

$$\kappa_u \geqq \kappa^* \qquad \text{or} \qquad \kappa_F^*\left(\frac{2\pi}{L^*} \text{ or } \frac{2\pi}{L_F^*}\right) \qquad (7.19)$$

In a stochastic finite element analysis where the material properties or boundary conditions are assumed to be stochastic, we face the determination of the finite element size corresponding to the randomness of the material properties in space. For this problem, the correlation scales may also be useful as a measure of the relationships between the element size and the material randomness in space.

8. CONCLUSIONS

In this report, we reinterpret the correlation scales previously defined in the literature from the viewpoint of the statistical analysis of observed field data. By considering the averaging process and the difference process, two typical definitions for correlation scales are consistently derived, and also new definitions for the same correlation scales are obtained which make it possible to estimate the correlation scales from variances easily calculated from observed field data. The statistical assessments of the estimation of correlation scales from the variances are also briefly presented.

By extending the procedure for one-dimensional stochastic processes to two-dimensional stochastic processes, the correlation scales (area) of two-dimensional processes are defined and visually illustrated using a digital simulation techique. An estimation procedure for these corelation scales for two-dimensional processes is presented.

Finally some new application samples of correlation scales defined and reinterpreted in this study are briefly presented. They are the applications of correlation scales into (1) the approximate distribution of the maximum values of stochastic processes over a finite length, (2) the estimation of the seismic ground rms (root mean square) strain, (3) a measure of the upper cut-off frequency in a digital time series analysis and simulation, and (4) a measure of the relationships between the finite element size and the material randomness in stochastic finite element analysis.

9. ACKNOWLEDGEMENT

This work was supported by the Tokyo Gas Co., Ltd., Tokyo, Japan. The authors gratefully acknowledge the above support.

REFERENCES

1. Batchelor, G.K., The Theory of Homogeneous Turbulence, (Cambridge: Cambridge University Press), 1953.

2. Bendat, J.S. and Piersol, A.G., Random Data: Analysis and Measurement Procedures, (NY: Wiley), 1971.

3. Cramer, H. and Leadbetter, M.R., Stationary and Related Stochastic Processes, (NY: Wiley), 1967.

4. Davneport, A.G., "Note on the Random Distribution of the Largest Value of a Random Function with Emphasis on Gust Loading," Proceedings of the Institute of Civil Engineers, London, Vol. 28, 1964.

5. Doob, J.L., Stochastic Processes, (NY: Wiley), 1953.

6. Gumbel, E.J., Statistics of Extremes, (NY: Columbia University Press), 1958.

7. Harada, T. and Shinozuka, "Ground Deformation Spectra," Proceedings of the 3rd US National Conference on Earthquake Engineering, Charleston, SC, August 24-28, 1986.

8. Lin, Y-K., Probabilistic Theory of Structural Dynamics, (NY: McGraw-Hill), 1967.

9. Lin, Y-K., Fujimori, Y. and Ariaratnam, S.T., "Rotor Blade Stability in Turbulent Flow," Journal of Aermican Institute of Aeronautics and Astronautics, Vol. 17, No. 16, 1979, pp. 545-552.

10. Lumley, J.L., Stochastic Tools in Turbulence, (NY: Academic Press), 1970.

11. Panchev, S., Random Functions and Turbulence, (Oxford: Pergamon Press), 1971.

12. Papoulis, A., Signal Analysis, (NY: McGraw-Hill), 1977.

13. Shinozuka, M. and Jan, C-M., "Digital Simulation of Random Processes and Its Applications," Journal of Sound and Vibration, Vol. 27, No. 1, 1972, pp. 111-128.

14. Stratonovich, R.L., Topics in the Theory of Random Noise, (NY: Gordon and Breach), 1967.

15. Tatarski, V.I., Wave Propagation in a Turbulent Medium, (NY: McGraw-Hill), 1967.

16. Taylor, G.I., "Diffusion by Continuous Movements," Proceedings of the London Mathematical Society, Vol. 20, 1921, pp. 196-222.

Monin, A.S. and Yaglom, A.M., Statistical Fluid Mechanics: Mechanics of

Turbulence, (Cambridge: The MIT Press), 1971.

17. Taylor, G.I. "Statical Theory of Turbulence," Proceedings of the Royal Society, Vol. 151, Series A, 1935, pp. 421-478.

18. Vanmarcke, E., Random Fields: Analysis and Synethesis, (Cambridge: The MIT Press), 1983.

19. Wu, W-F., "Some Closure Problems in Stochastic Dynamic of Solids," Ph.D. Thesis, Dept. of Aeronatucial and Astronautical Engineering, University of Illinois, 1985.

20. Yaglom, A.M, An Introduction to the Theory of Stationary Random Functions, (NY: Dover), 1973.

Table 1-1 Summary of Definitions for The Scale of Correlation

Items	Definitions	Authors
A	$\frac{1}{\sigma^2}\int_{-\infty}^{\infty} R(\xi)\, d\xi$	Taylor (1935), Batchelor (1953), Tatarski (1961), Monin and Yaglom (1965), Panchev (1971), Bendat and Piesol (1971), Lumley (1970), Vanmarcke (1983)
B	$\frac{1}{\sigma^2}\int_{-\infty}^{\infty} \lvert R(\xi)\rvert\, d\xi$	Stratonovich (1967)
C	$\sqrt{-\frac{2R(0)}{R''(0)}}$	Tatarski (1961), Monin and Yaglom (1965), Lumley (1970), Harada and Shinozuka (1985)
D	$\dfrac{\int_0^{\infty} \xi \lvert R(\xi)\rvert\, d\xi}{\int_0^{\infty} \lvert R(\xi)\rvert\, d\xi}$	Lin, Fujimori and Ariaratnam (1979)

Note: $\sigma^2 = R(0)$ = Variance, $R(\xi)$ = Correlation Function, $R''(\xi) = d^2R(\xi)/d\xi^2$

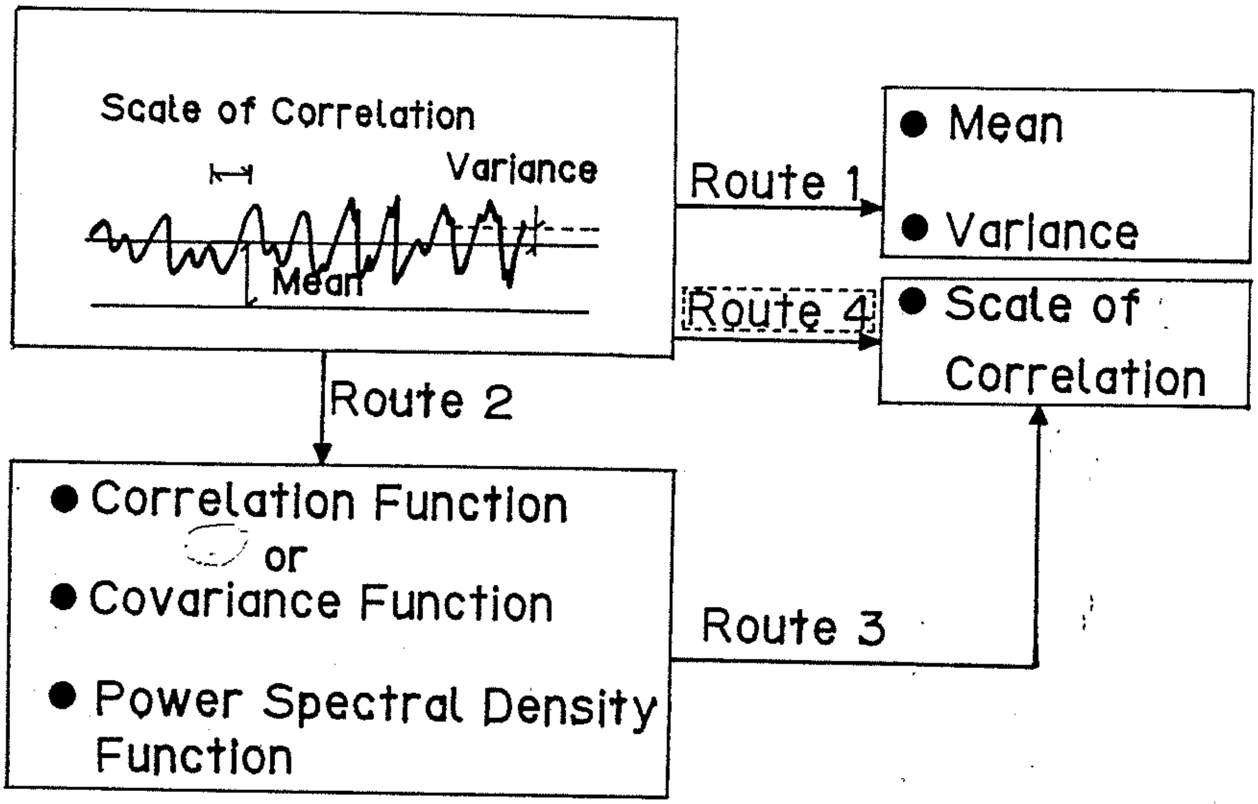


Fig. 1-1 Schematic Diagram Showing Relationship Between Stochastic Process And Its Statistics

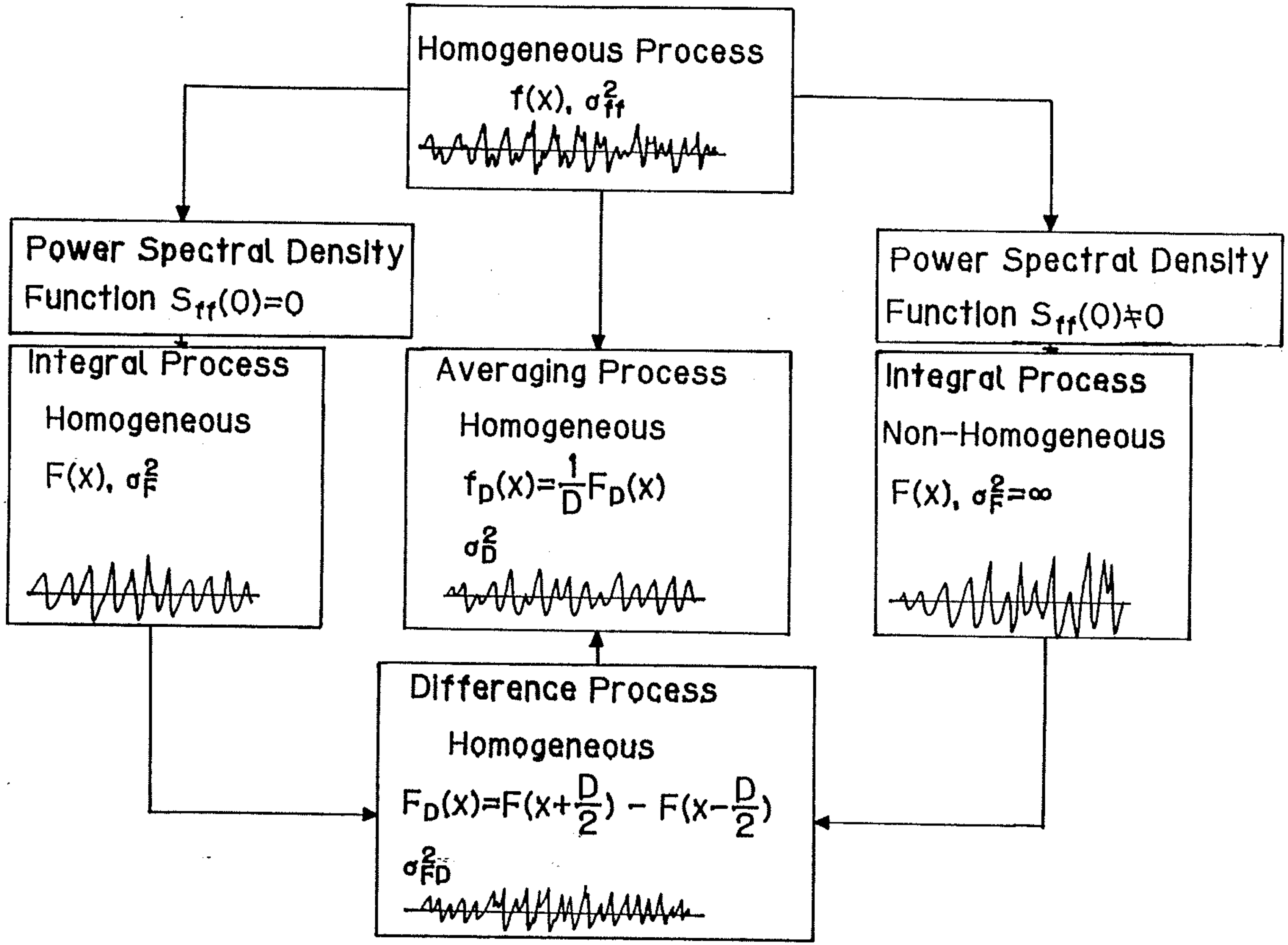
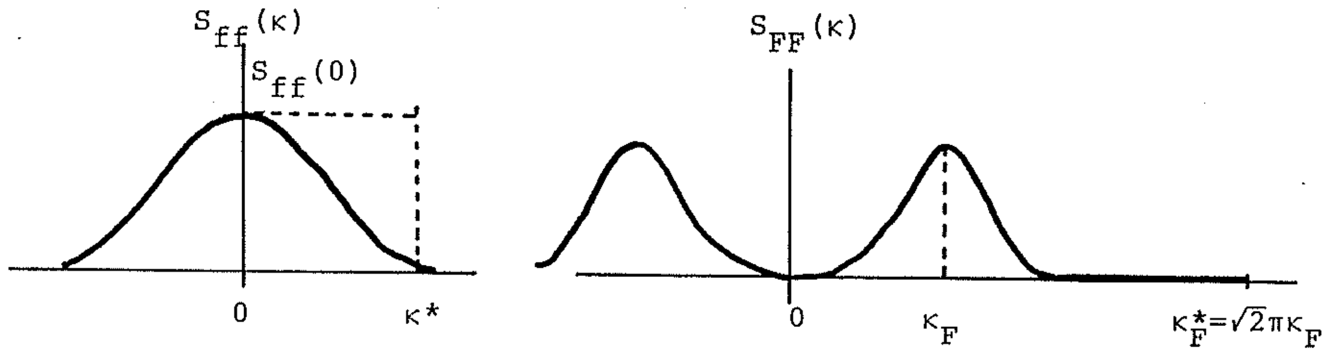


Fig. 2-1 Schematic Diagram Showing Relationship Among Integral Process, Averaging Process And Difference Process



(a) Case I

(b) Case II

Fig. 3-1 Schematical Illustration of Significance of The Spectral Scale

4-1

Table 1 AUTO-CORRELATION AND SPECTRAL DENSITY FUNCTIONS

Type	$R(\xi)$	$S(\kappa)$
1	$\frac{b^2}{\xi^2+b^2}$	$\frac{1}{2\cdot 0!}\, be^{-b\lvert\kappa\rvert}$
2	$\frac{b^4(b^2-3\xi^2)}{(\xi^2+b^2)^3}$	$\frac{1}{2\cdot 2!}\, b^3\kappa^2 e^{-b\lvert\kappa\rvert}$
3	$\frac{b^6(b^4-10b^2\xi^2+5\xi^4)}{(\xi^2+b^2)^5}$	$\frac{1}{2\cdot 4!}\, b^5\kappa^4 e^{-b\lvert\kappa\rvert}$
4	$\frac{b^8(b^6-21b^4\xi^2+35b^2\xi^4-7\xi^6)}{(\xi^2+b^2)^7}$	$\frac{1}{2\cdot 6!}\, b^7\kappa^6 e^{-b\lvert\kappa\rvert}$
5	$\frac{b^{10}(b^8-36b^6\xi^2+126b^4\xi^4-84b^2\xi^6+9\xi^8)}{(\xi^2+b^2)^9}$	$\frac{1}{2\cdot 8!}\, b^9\kappa^8 e^{-b\lvert\kappa\rvert}$
6	$\frac{b^{12}}{(\xi^2+b^2)^{11}} \times (b^{10}-55b^8\xi^2+330b^6\xi^4-462b^4\xi^6+$ $+\, 165b^2\xi^8-11\xi^{10})$	$\frac{1}{2\cdot 10!}\, b^{11}\kappa^{10} e^{-b\lvert\kappa\rvert}$

4-2

Table 2 AUTO-CORRELATION AND SPECTRAL DENSITY FUNCTIONS

Type	$R(\xi)$	$S(\kappa)$
1	$e^{-\left(\frac{\xi}{b}\right)^2}$	$\frac{1}{2}\cdot\frac{b}{\sqrt{\pi}}\exp[-\left(\frac{b\kappa}{2}\right)^2]$
2	$[1-2(\frac{\xi}{b})^2]e^{-\left(\frac{\xi}{b}\right)^2}$	$\frac{1}{2}\cdot\frac{b^3}{2\sqrt{\pi}}\,\kappa^2\exp[-\left(\frac{b\kappa}{2}\right)^2]$
3	$[1-4(\frac{\xi}{b})^2+\frac{4}{3}(\frac{\xi}{b})^4]e^{-\left(\frac{\xi}{b}\right)^2}$	$\frac{1}{2}\cdot\frac{b^5}{12\sqrt{\pi}}\,\kappa^4\exp[-\left(\frac{b\kappa}{2}\right)^2]$
4	$[1-6(\frac{\xi}{b})^2+4(\frac{\xi}{b})^4-\frac{8}{15}(\frac{\xi}{b})^6]e^{-\left(\frac{\xi}{b}\right)^2}$	$\frac{1}{2}\cdot\frac{b^7}{120\sqrt{\pi}}\,\kappa^6\exp[-\left(\frac{b\kappa}{2}\right)^2]$
5	$[1-8(\frac{\xi}{b})^2+8(\frac{\xi}{b})^4-\frac{32}{15}(\frac{\xi}{b})^6+\frac{16}{105}(\frac{\xi}{b})^8]e^{-\left(\frac{\xi}{b}\right)^2}$	$\frac{1}{2}\cdot\frac{b^9}{1680\sqrt{\pi}}\,\kappa^8\exp[-\left(\frac{b\kappa}{2}\right)^2]$
6	$[1-10(\frac{\xi}{b})^2+\frac{40}{3}(\frac{\xi}{b})^4-\frac{16}{3}(\frac{\xi}{b})^6+\frac{16}{21}(\frac{\xi}{b})^8-\frac{32}{945}(\frac{\xi}{b})^{10}]e^{-\left(\frac{\xi}{b}\right)^2}$	$\frac{1}{2}\cdot\frac{b^{11}}{30240\sqrt{\pi}}\,\kappa^{10}\exp[-\left(\frac{b\kappa}{2}\right)^2]$

Table 4-3 Examples of Family of Correlation and Spectral Density Functions Where $S(0) \neq 0$

Type	$R(\xi)$	$S(\kappa)$
1	$1 - \frac{\lvert\xi\rvert}{b} \quad \lvert\xi\rvert \leqq b$ $0 \quad$ otherwise	$\frac{2}{\pi b} \frac{(1 - \cos b\kappa)}{\kappa^2}$
2	$e^{-(\frac{\lvert\xi\rvert}{b})}$	$\frac{b}{\pi} \frac{1}{(b\kappa)^2 + 1}$
3	$[1 + \frac{\lvert\xi\rvert}{b}]e^{-(\frac{\lvert\xi\rvert}{b})}$	$\frac{2b}{\pi} \frac{1}{[(b\kappa)^2 + 1]^2}$
4	$[1 + \frac{\lvert\xi\rvert}{b} + \frac{1}{3}(\frac{\xi}{b})^2]e^{-(\frac{\lvert\xi\rvert}{b})}$	$\frac{8b}{3\pi} \frac{1}{[(b\kappa)^2 + 1]^3}$
5	$e^{-(\frac{\xi}{b})^2}$	$\frac{1}{2} \frac{b}{\sqrt{\pi}} e^{-(\frac{b\kappa}{2})^2}$
6	$\frac{1}{1 + (\frac{\xi}{b})^2}$	$\frac{1}{2} b \; e^{-b\lvert\kappa\rvert}$

Table 4-4 The Scale of Correlation for Various Correlation Functions of f(x)

Type in Table 1	L^*	L_F^*	Type in Table 2	L^*	L_F^*	Type in Table 3	L^*	L_F^*
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1 [I]*	πb	(b)**	1 [I]	$\sqrt{\pi}b$	(b)	1 [I]	b	0
2 [II]	0	$\frac{b}{\sqrt{6}}$	2 [II]	0	$\frac{b}{\sqrt{3}}$	2 [I]	2b	0
3 [II]	0	$\frac{b}{\sqrt{15}}$	3 [II]	0	$\frac{b}{\sqrt{5}}$	3 [I]	4b	($\sqrt{2}b$)
4 [II]	0	$\frac{b}{\sqrt{28}}$	4 [II]	0	$\frac{b}{\sqrt{7}}$	4 [I]	$\frac{16b}{3}$	($\sqrt{6}b$)
5 [II]	0	$\frac{b}{\sqrt{45}}$	5 [II]	0	$\frac{b}{\sqrt{9}}$	5 [I]	$\sqrt{\pi}b$	(b)
6 [II]	0	$\frac{b}{\sqrt{66}}$	6 [II]	0	$\frac{b}{\sqrt{11}}$	6 [I]	πb	(b)

Note: * I and II in [ ] mean Case I and Case II.
** The scale of correlation in ( ) is for correlation function of F(x).

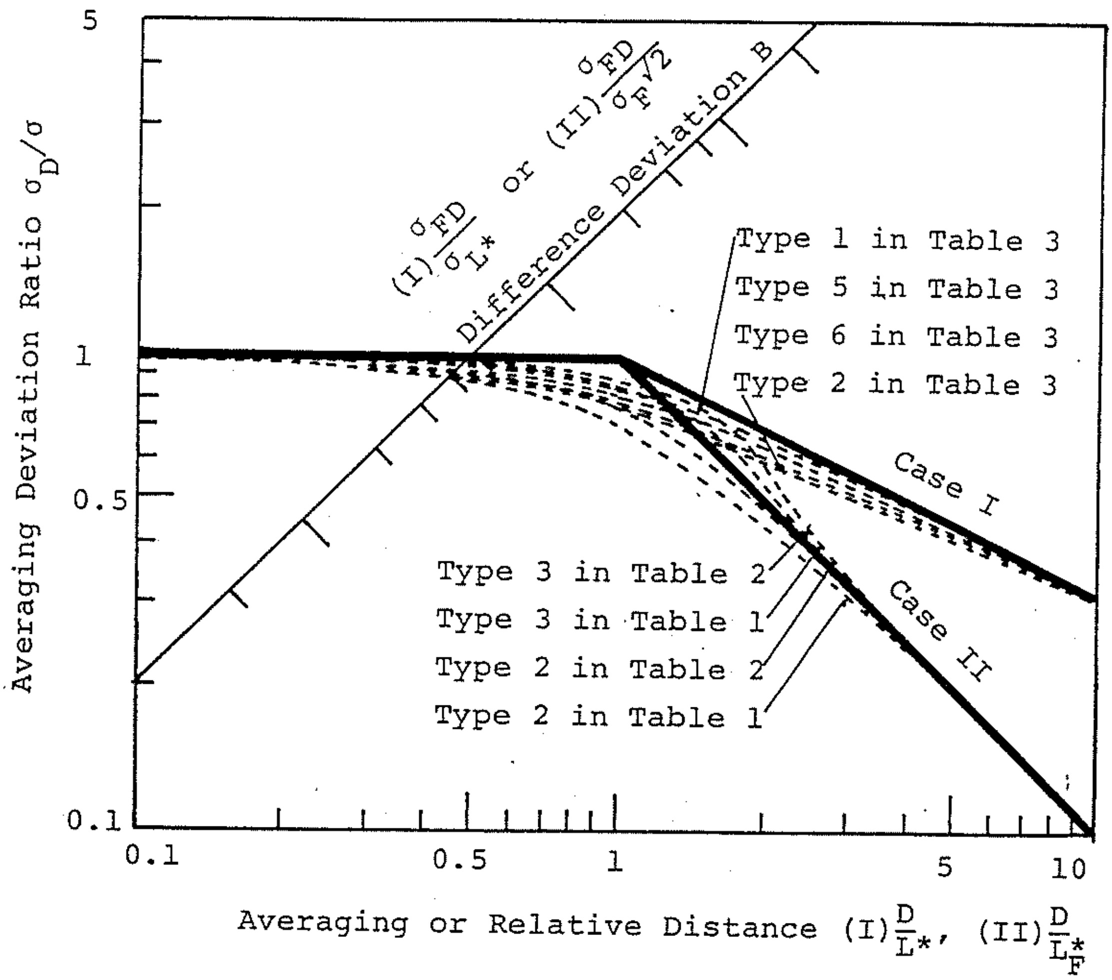


Fig. 4-1 σ_D-D Diagram

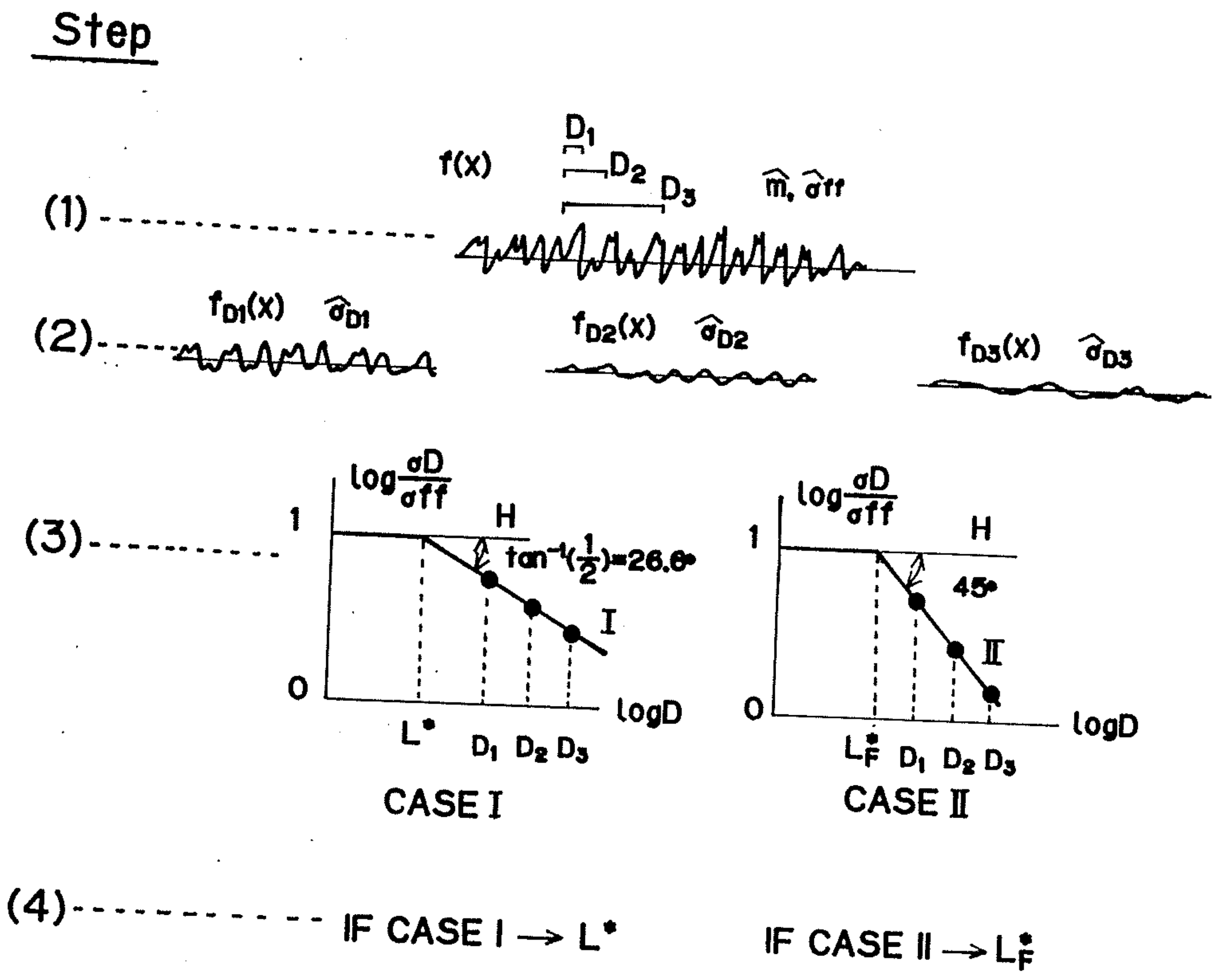


Fig. 5-1 Estimation Procedure of The Scale of Correlation

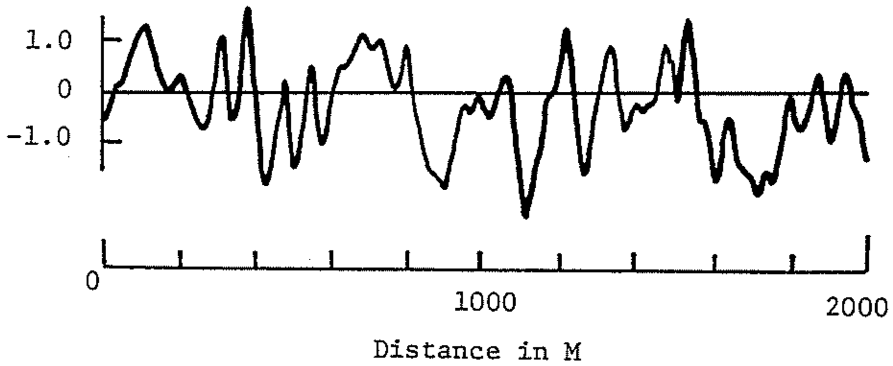


Fig. 5-2 Sample Function of f(x)

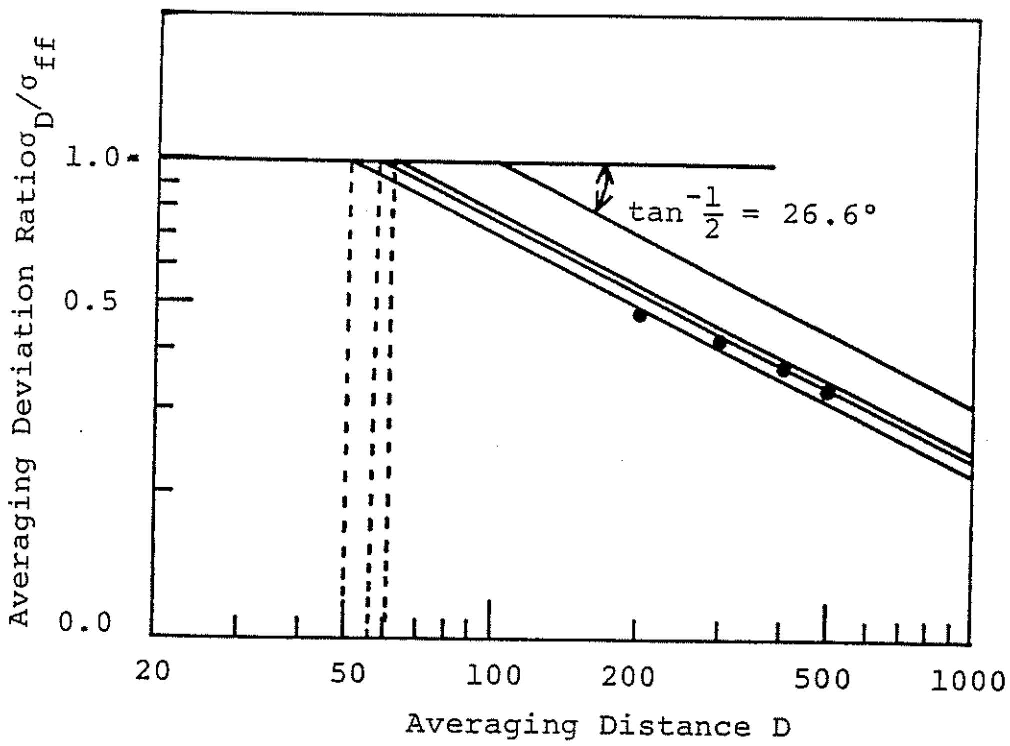


Fig. 5-3 Numerical Example for Graphical Estimation of The Scale of Correlation

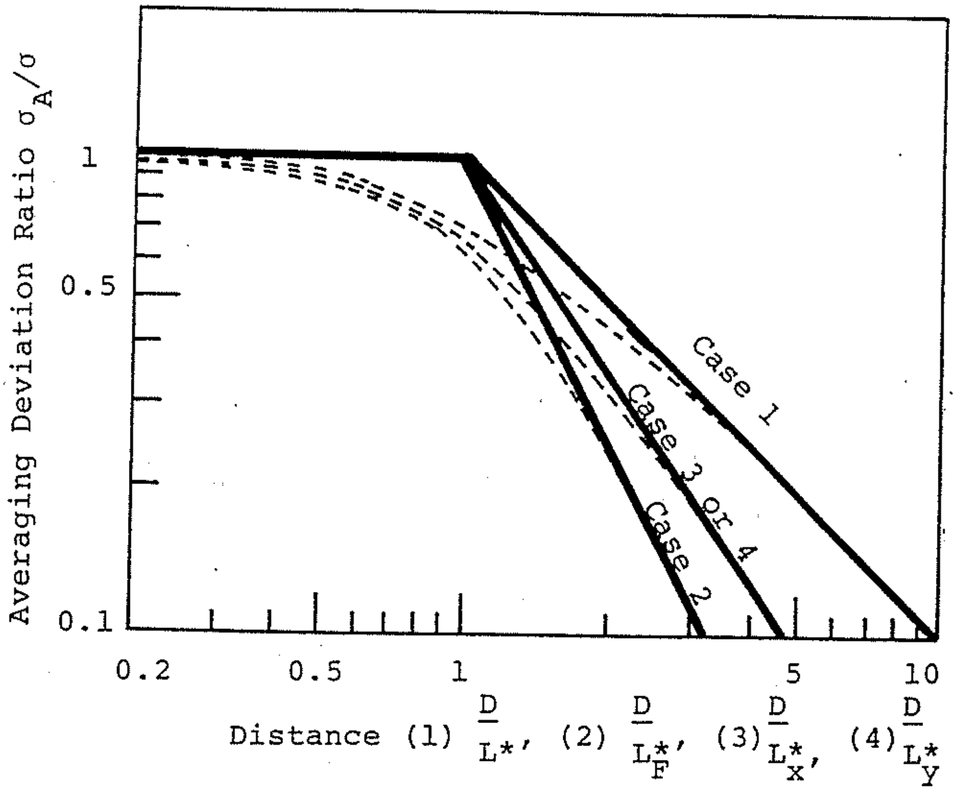


Fig. 6-1 σ_A-D Diagram for Two-Dimensional Process

Where $D=D_x=D_y$

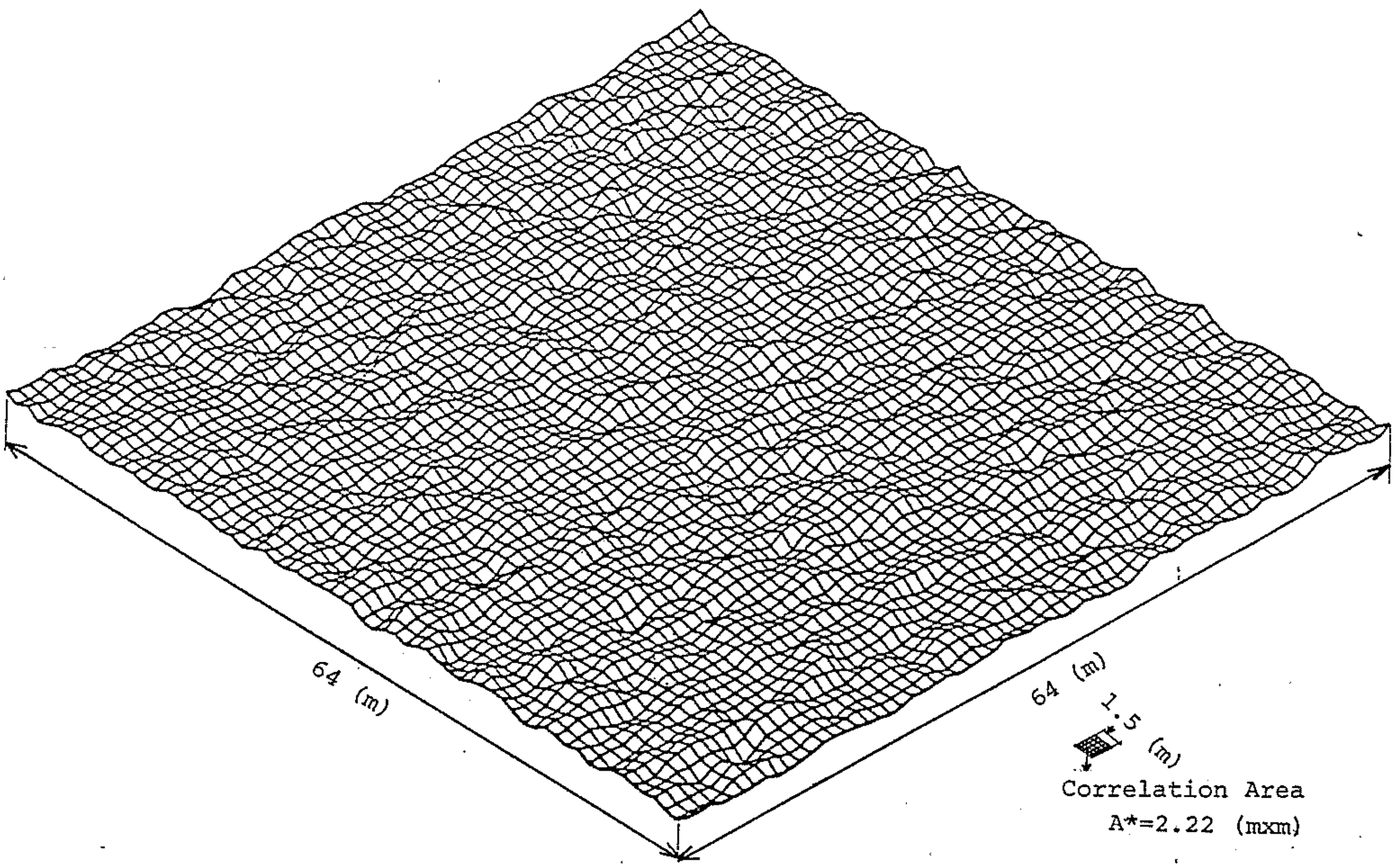


Fig. 6-2 Sample Function of f(x,y) for Case 1

Frame : 4
File : USERDISK1:[YAMAZAKI.R]NCAR.PLT;1
Printed : 26-MAR-86 17:50:18

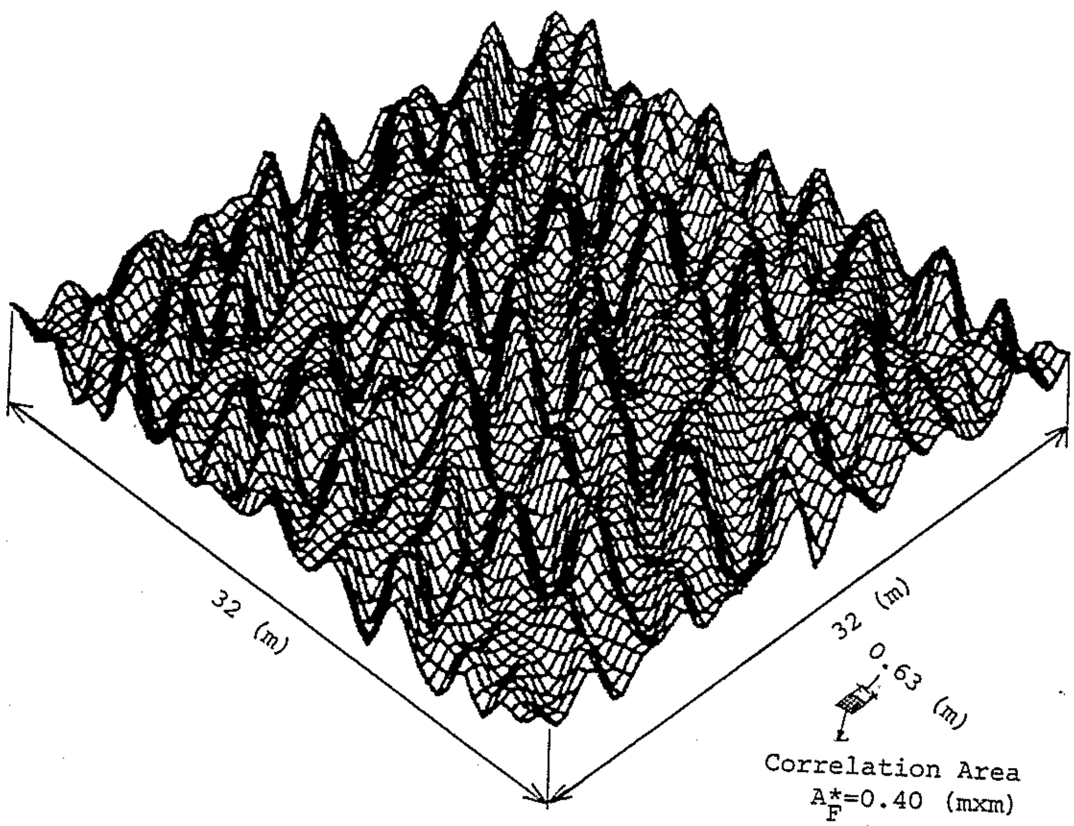


Fig. 6-3 Sample Function of f(x,y) for Case 2

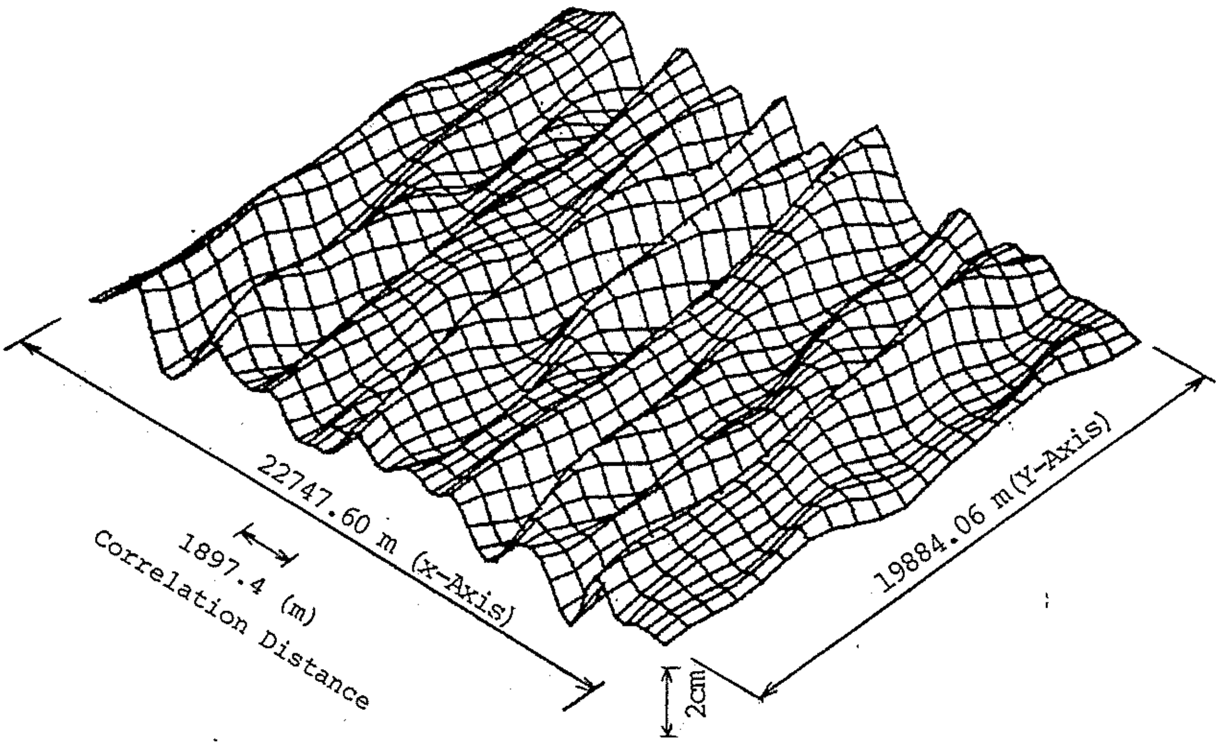


Fig. 6-6 Sample Function of f(x,y) for Case 3

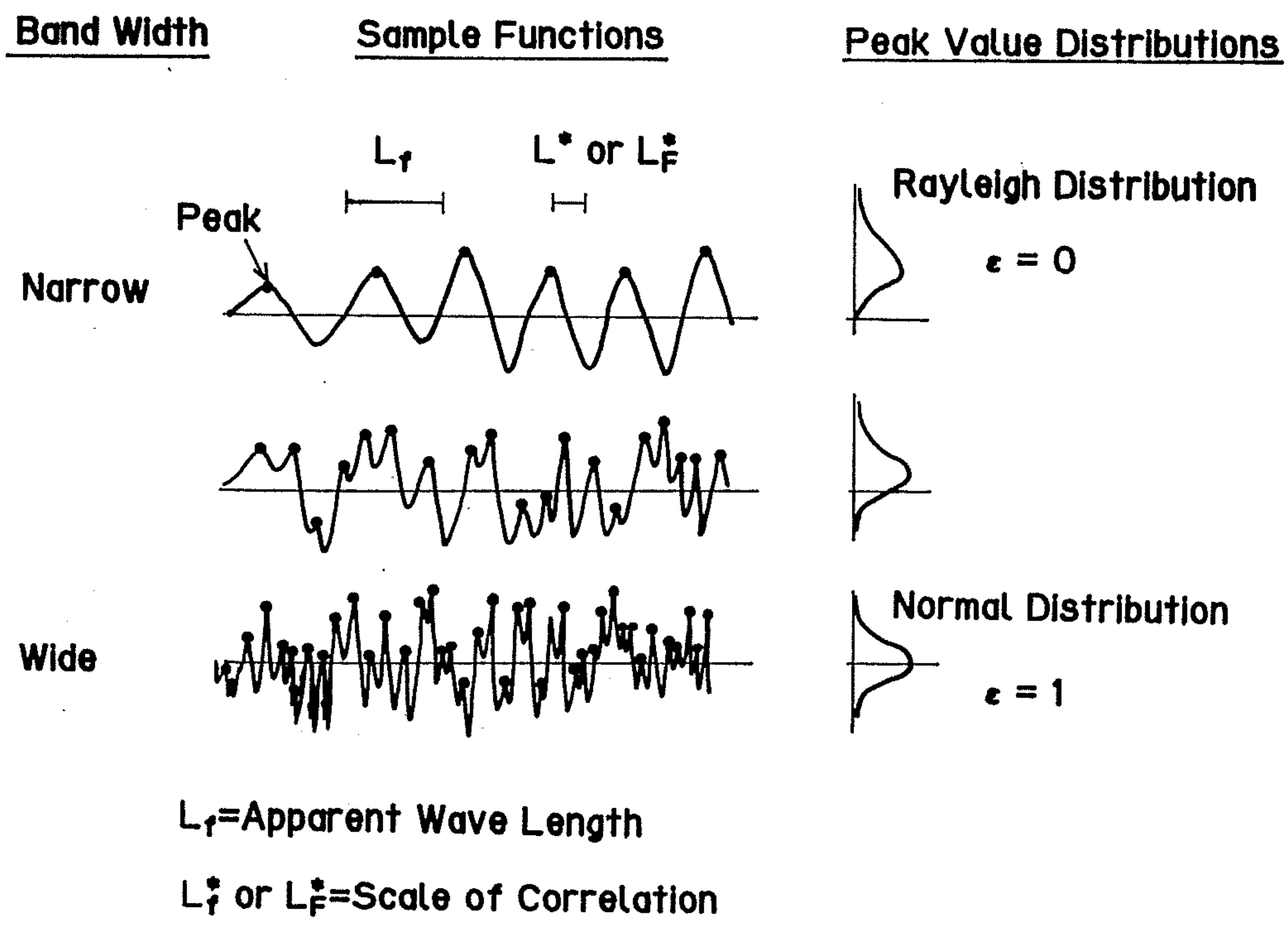


Fig. 7-1 Schematic Illustration of Sample Functions and Peak Value Distributions

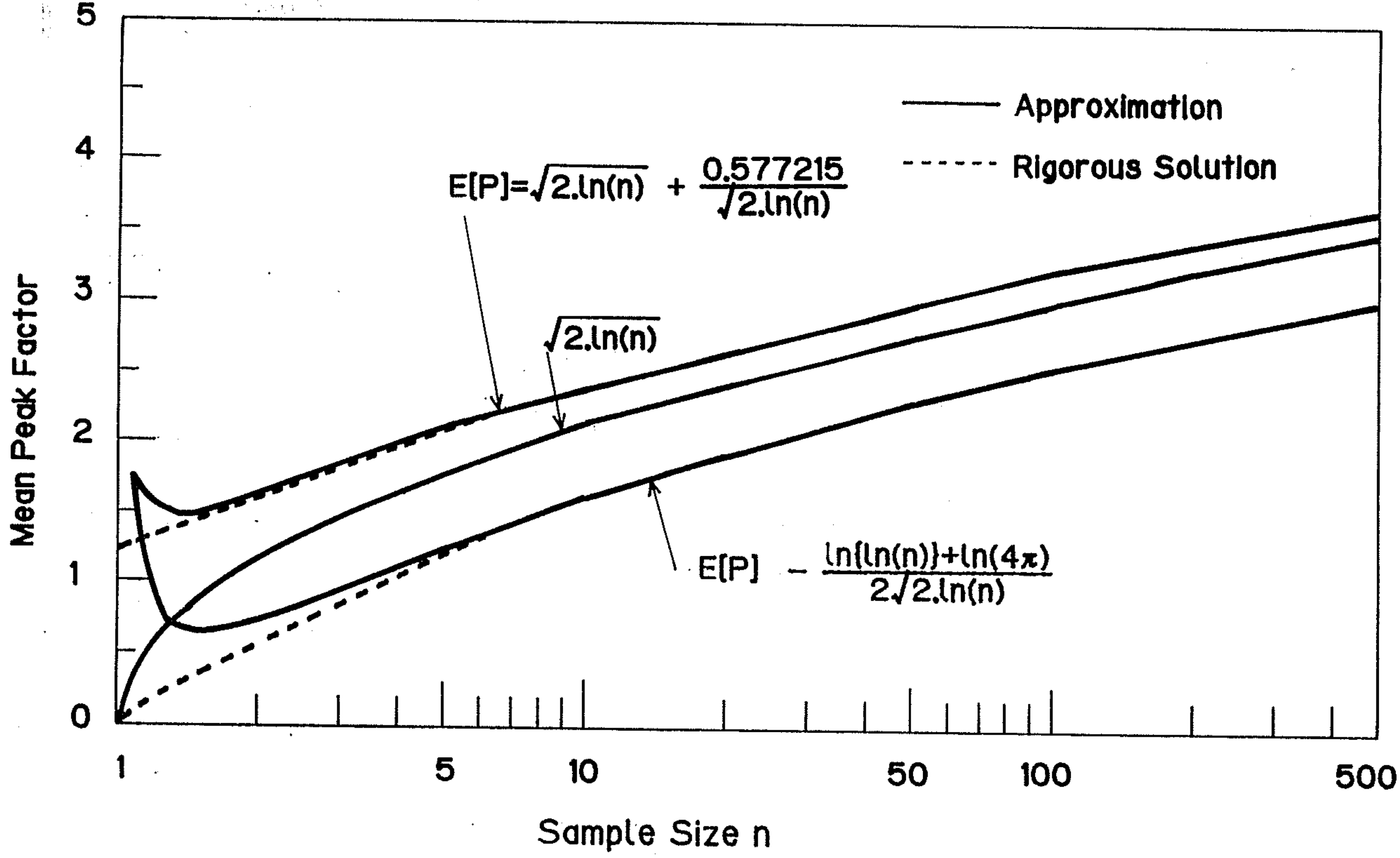


Fig.7-2 Relationships BetweenMean Peak Factor And Independent Sample Size